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A REPORT

ON

A THEORY OF PERMANENTS

PART II: CIRCULANT PERMANENTS

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Summary

Introduction

In this part, we give a detailed discussion of the Method of Links for determining the coefficients in the expansion of a circulant permanent. Although this method will prove satisfactory for many types of terms in such an expansion, there will be many others for which the use of links would involve a prohibitive amount of work by hand. Other methods may be used; two of these are discussed briefly in the last sections.

It should be noted that any given method is best suited for a certain kind of term.

The use of a machine would be desirable at certain points in the calculations. These places have been pointed out.

In the summary, there is gathered together a large number of formulas for coefficients of various kinds. These represent, in general, the cases where the idea of links can be used to advantage.

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PART II.

Circulant Permanents

I. Introduction.

In this Part we consider in greater detail the problem of obtaining the expansion of a circulant permanent. (See Part I, Chapter 2, section 5).

We give first a brief review of the previous discussion (of Part I).

A circular permanent is defined by

$$(1.1)\qquad C_n = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ a_2 & a_3 & a_4 & \cdots & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{vmatrix}^{+}$$

If any term in the expansion of C_n be denoted by

$$(1.2)\qquad A a_0^{e_0} a_1^{e_1} a_2^{e_2} \cdots\ a_{n-1}^{e_{n-1}}$$

then its weight $w \equiv 0 \pmod n$, i.e.

$$(1.3)\quad w = e_1 + 2e_2 + 3e_3 + \cdots + (n-1)e_{n-1} \equiv 0 \pmod n.$$

The exponents e_i must also satisfy the relation

$$(1.4)\qquad e_0 + e_1 + e_2 + \cdots + e_{n-1} = n.$$

Two basic properties of the terms are:

(1) If $A a_0^{e_0} a_1^{e_1} \cdots a_{n-1}^{e_{n-1}}$ is a term of C_n then so also is

$Aa_p^{e_o} a_{p+1}^{e_1} \cdots a_{p-1}^{e_{n-1}}$ <u>for all values of p</u>.

(Cyclic permutation of subscripts)

(2) <u>If</u> $Aa_o^{e_o} a_1^{e_1} \cdots a_{n-1}^{e_{n-1}}$ <u>is a term of C_n then so also is</u>

$Aa_o^{e_o} a_m^{e_1} a_{2m}^{e_2} \cdots a_{(n-1)m}^{e_{n-1}}$ <u>where m is prime to n</u>. (Subscripts taken mod n).

We know that C_n is the coefficient of $x_o x_1 x_2 \cdots x_{n-1}$ in the expansion of

(1.5) $$\prod_{h=o}^{n-1} (a_{n-h} x_o + a_{n-h+1} x_1 + \cdots + a_{n-h-1} x_{n-1})$$

If $i_o i_1 i_2 \cdots i_{n-1}$ be a permutation of $0,1,2, \ldots,(n-1)$, then a term of the above product-expansion is

$$a_{i_o} a_{i_1-1} a_{i_2-2} \cdots a_{i_{n-1}-(n-1)} x_{i_o} x_{i_1} \cdots x_{i_{n-1}}$$

where x_{i_p} comes from the $(p+1)^{st}$ factor.

One method to obtain the coefficient A of $Aa_o^{e_o} a_1^{e_1} \cdots a_{n-1}^{e_{n-1}}$ is as follows:

Let $k_o, k_1, k_2, \ldots, k_{n-1}$ be a permutation of the n numbers (the subscripts of the term) $\underbrace{o, \ldots, o,}_{e_o}$ $\underbrace{1, \ldots, 1,}_{e_1}$ $\underbrace{2 \ldots 2,}_{e_2}$ $\ldots, \underbrace{n-1, \ldots, n-1,}_{e_{n-1}}$ and consider the congruences

(1.6) $$i_o \equiv k_o,\ i_1-1 \equiv k_1,\ i_2-2 \equiv k_2,\ \ldots,\ i_{n-1}-(n-1) \equiv k_{n-1}, \quad (\text{mod } n).$$

If (1.6) has a solution $i_o, i_1, \ldots, i_{n-1}$ <u>which is a permutation of 0, 1, 2, ..., n-1</u> then these values of the i's will give a term $a_o^{e_o} a_1^{e_1} \cdots a_{n-1}^{e_{n-1}}$ whose coefficient A will <u>equal the number of distinct</u>

solutions of (1.6).

Note also that the particular permutation $i_o i_1 i_2 \ldots i_{n-1}$ gives the order in picking out the elements from C_n. Thus, the solution $(i_o i_1 \ldots i_{n-1})$ means to pick a_{i_o} from row 1, a_{i_1-1} from row 2, etc.

The correspondence between the $k_o k_1 \ldots k_{n-1}$ and $i_o i_1 \ldots i_{n-1}$ is conveniently represented by the form:

$$\begin{array}{lllllll} N: & 0 & 1 & 2 & \ldots & n-1 & (= \text{normal order}) \\ k: & k_o & k_1 & k_2 & \ldots & k_{n-1} & (= k\text{-permutation}) \\ i: & i_o & i_1 & i_2 & \ldots & i_{n-1} & (= i\text{-permutation}), \end{array}$$

and from (1.6) we have

$$(1.7) \qquad i_t \equiv t + k_t \ , \ (t = 0, 1, \ldots, n-1)$$

We define an <u>i-permutation</u> as any permutation of 0, 1, 2, ..., n-1, and a <u>k-permutation</u> as a permutation $k_o k_1 \ldots k_{n-1}$ which produces an i-permutation by (1.7).

Three important properties of k-permutations are:

(1) Any cyclic permutation of a k-permutation will also be a k-permutation (but the associated i-permutation may not be distinct from that of the given k-permutation).

(2) If $k_o k_1 k_2 \ldots k_{n-1}$ is a k-permutation then so also is $(k_o + x)(k_1 + x) \ldots (k_{n-1} + x)$ for $x = 0, 1, \ldots, n-1$.

(3) If $k_o k_1 \ldots k_{n-1}$ is a k-permutation then so also is $(mk_o)(mk_1) \cdots (mk_{n-1})$ where m is prime to n provided the elements mk_i are rearranged according to the order determined by $mN \equiv 0, m, 2m, \ldots$

To illustrate this last property consider

N:	0 1 2 3 4 5 6 7 8 9
k:	1 2 8 2 8 3 0 7 9 0
i:	1 3 0 5 2 8 6 4 7 9

with $m = 3$:

mN:	0 3 6 9 2 5 8 1 4 7	
mk:	3 6 4 6 4 9 0 1 7 0	
(mk)':	3 1 4 6 7 9 4 0 0 6	(= mk rearranged)
(mi)':	3 2 6 9 1 4 0 7 8 5	

Their proofs are evident.

Suppose we wish to determine the coefficient A of the term $a_0^2 a_2^2$ of a C_4. Here there are four subscripts 0, 0, 2, 2 from which the k-permutations are to be constructed. Due to property (1) we may take $k_0 = 0$, giving 3 possibilities:

N:	0 1 2 3	0 1 2 3	0 1 2 3
k:	0 0 2 2	0 2 0 2	0 2 2 0
i:	0 1 0 1	0 3 2 1	0 3 0 3

Hence 0202 is a k-permutation with 0321 its associated i-permutation. Only 2 of the four cyclic permutations of 0202 give distinct i-permutations:

N:	0 1 2 3	0 1 2 3
k:	0 2 0 2	2 0 2 0
i:	0 3 2 1	2 1 0 3

It follows that the coefficient $A = 2$.

It should be kept in mind that a k-permutation is composed of the n subscripts of any particular term and the associated i-permutation determines the manner of picking that term from the rows of C_n.

2. Links and Chains.

One method for the calculation of the coefficients A of terms of C_n is based on the idea of a link. We give a discussion of this idea here.

Consider the term $a_0^2 a_1 a_2^2 a_3 a_7 a_8^2 a_9$ of a C_{10} and the following four associated k- and i-permutations:

	(a)	(b)	(c)	(d)
N:	0123456789	0123456789	0123456789	0123456789
k:	1282830790	3081802297	1207392880	1972288003
i:	1305286479	3104258976	1320748569	1095634782

Write the i-permutations in cycle form (omitting 1-cycles). This gives:

(a) i = (01358742)
(b) i = (0342)(6879)
(c) i = (013)(475)(68)
(d) i = (01)(29)(35)(46)

Now rearrange the N and i rows in the order determined by the cycles. For case (c) this gives:

N:	0 1 3	4 7 5	6 8
k:	1 2 7	3 8 9	2 8
i:	1 3 0	7 5 4	8 6

The resulting sequences in the k-row: (1, 2, 7), (3, 8, 9), (2, 8) are called <u>links</u>. Every i-permutation can thus be put in correspondence with a set of links. For the above four examples we obtain the links:

(a) (1, 2, 2, 3, 9, 7, 8, 8)
(b) (3, 1, 8, 8)(2, 9, 2, 7)
(c) (1, 2, 7)(3, 8, 9)(2,8)
(d) (1, 9)(2, 8)(2, 8)(3, 7)

Each such sequence of links will be called a <u>chain</u>. The number of links in a chain will equal the number of cycles (omitting one-cycles) of the corresponding i-permutation. The length of a link equals that of its corresponding cycle. A chain of t links is called a t-link chain or a t-chain for short. Since a k-permutation gives rise to an i-permutation we may speak of a k-permutation as being a 1-chain, or a 2-chain case, etc.

The above four examples are respectively 1-chain, 2-chain, 3-chain, and 4-chain. (Note that 0 never occurs as an element of a link).

Corresponding to a particular term of C_n we have a variety of i-permutations to each of which we associate some t-chain. It is evident that all these various t-chains are composed of the same elements, these elements being in fact the non-zero subscripts of the term. For the term mentioned above (of a C_{10}) the elements would be 1, 2, 2, 3, 7, 8, 8, 9. The value of t may of course vary from one i-permutation to another for a given term (as illustrated by the 4 cases of our example).

To obtain the characteristic properties of a link we take a general cycle of an i-permutation $(s_0 s_1 s_2 \ldots s_{j-1})$:

$$\begin{array}{lccccc} N: & s_0 & s_1 & s_2 & \cdots & s_{j-1} \\ k: & p_1 & p_2 & p_3 & \cdots & p_j \\ i: & s_1 & s_2 & s_3 & \cdots & s_0 \end{array}$$

The link is $(p_1, p_2, p_3, \ldots, p_j)$, and $s_1 = s_0 + p_1$, $s_2 = s_1 + p_2$, $\ldots$, $s_0 = s_{j-1} + p_j$. Hence $p_1 = s_1 - s_0$, $p_2 = s_2 - s_1$, $\ldots$, $p_j = s_0 - s_{j-1}$, so that

(2.1) $$p_1 + p_2 + \cdots + p_j \equiv 0 \pmod n$$

(2.2) $$p_r + p_{r+1} + \cdots + p_s \not\equiv 0$$

(for all r, s except $r = 1$, $s = j$; $r < s$).

This gives us the definition:

A link (mod n) is an ordered sequence of numbers $(p_1, p_2, p_3, \ldots, p_j)$ such that

(1) $\sum_{i=1}^{j} p_i \equiv 0 \pmod n$ (no $p_i = 0$)

(2) $\sum_{i=r}^{s} p_i \not\equiv 0 \pmod{n}$, $(r < s$; exclude $r = 1$, $s = j)$

We have seen that the coefficient A of any term of C_n is equal to the number of k-permutations producing distinct i-permutations. Recall that a k-permutation is some permutation of the (non-zero) subscripts of a term. Since to every i-permutation there corresponds some t-link chain and every link is composed of (non-zero) subscripts, the totality of links containing all such subscripts, it follows that <u>to obtain all possible k-permutations it is sufficient to use only those subscript permutations which can be decomposed into links</u>. Such subscript-permutations will be called <u>link-permutations</u>.

To a given t-chain there may correspond several i-permutations. Take for example the case (b) above with the 2-link chain (3, 1, 8, 8)(2, 9, 2, 7). Place the first link starting at 0 of N:

N:	0	1	2	3	4	5	6	7	8	9
k:	3		8	1	8					
i:	3		0	4	2					

After placing the first link there are 6 available places left to start the second link. Suppose we start at position <u>a</u> of N:

N:	a	$a+2$	$a+1$	$a+3$
k:	2	9	2	7
i:	$a+2$	$a+1$	$a+3$	a

Therefore <u>a</u> must satisfy the conditions $a \not\equiv 0,3,4,2$; $a+2 \not\equiv 0,3,4,2$; $a+1 \not\equiv 0,3,4,2$; $a+3 \not\equiv 0,3,4,2$,

and the only solutions are $a = 5$, $a = 6$, giving:

	a = 5	a = 6
N:	0 1 2 3 4 5 6 7 8 9	0 1 2 3 4 5 6 7 8 9
k:	3 0 8 1 8 2 2 9 7 0	3 0 8 1 8 0 2 2 9 7
i:	3 1 0 4 2 7 8 6 5 9	3 1 0 4 2 5 8 9 7 6

Each of the 2 resulting k-permutations can be permuted cyclically for 10 positions and this will produce 20 i-permutations. Hence <u>the chain (3, 1, 8, 8)(2, 9, 2, 7) would give a contribution of 20 to the coefficient of the term</u> ($a_0^2 a_1 a_2^2 a_3 a_7 a_8^2 a_9$ of C_{10}).

The above discussion shows that the problem of determining the coefficient A of any given term of C_n may be reduced to two steps:
(1) <u>Determination of all link-permutations of the (non-zero) subscripts of the term, (this will give the chains</u>).
(2) <u>Determination of the number of all distinct i-permutations corresponding to each chain of step (1)</u>.

To simplify the calculations in these steps some general properties of t-link chains are needed. These are proved in the next section.

3. <u>Properties of Links and Chains</u>. We have seen that a given chain produces a set of k-permutations. We now determine what transformations on the links of a chain will leave this set of k-permutations invariant. Two chains so related (producing the same set of k-permutations) will be called <u>equivalent</u>.

We see first that <u>any cyclic permutation of the elements of a link gives another link</u>. This follows from the definition.

Consider now a general t-link chain

(3.1) $$(x_1, x_2, \cdots, x_p)(y_1, y_2, \cdots, y_q)(z_1, z_2, \cdots, z_r) \cdots$$

where there are t sets of parentheses. In the N, k, i form (3.1) is represented by

$$
(3.2)\quad
\begin{array}{llll}
N: & s_o s_1 \cdots s_{p-1} & t_o t_1 \cdots t_{q-1} & u_o u_1 \cdots u_{r-1} \cdots \\
k: & x_1 x_2 \cdots x_p & y_1 y_2 \cdots y_q & z_1 z_2 \cdots z_r \quad \cdots \\
i: & s_1 s_2 \cdots s_o & t_1 t_2 \cdots t_o & u_1 u_2 \cdots u_o \quad \cdots
\end{array}
$$

and $x_{h+1} = s_{h+1} - s_h$, $y_{h+1} = t_{h+1} - t_h$, etc.

The starting point s_o (in N-row) of the first chain is arbitrary, but t_o, u_o, ... must be chosen properly.

A convenient equivalent form to (3.2) is given by

$$
(3.2)\quad P \equiv \begin{bmatrix} s_o & s_o + x_1 & s_o + x_1 + x_2 \cdots & s_o + x_1 + \cdots + x_{p-1} \\ x_1 & x_2 & x_3 \quad \cdots & x_p \end{bmatrix} \begin{bmatrix} a & a+y_1 & a+y_1+y_2 \cdots \\ y_1 & y_2 & y_3 \quad \cdots \end{bmatrix}
$$

$$
\cdot \begin{bmatrix} b & b + z_1 \cdots \\ z_1 & z_2 \quad \cdots \end{bmatrix} \cdots
$$

(the i-cycle is $i = (s_o \;\; s_o+x_1 \;\; s_o+x_1+x_2 \cdots)(a \;\; a+y_1 \cdots)(b \;\; b+z_1 \cdots)\cdots$) where the top row of P represents the N-row elements and the bottom row the link-elements or k-row elements. (The i-row is omitted). Here a, b, ··· are a set of parameters which must satisfy the conditions.

$$
(3.3)\quad
\begin{cases}
a + (y_1 + y_2 + \cdots + y_h) \not\equiv s_o + (x_1 + x_2 + \cdots + x_g) \\
b + (z_1 + \cdots + z_f) \not\equiv s_o + (x_1 + \cdots + x_g) \\
b + (z_1 + \cdots + z_f) \not\equiv a + (y_1 + \cdots + y_h)
\end{cases}
$$

etc.

and $h = 0, 1, 2, \ldots, q-1$; $g = 0, 1, \ldots, p-1$; $f = 0, 1, \ldots, r-1$; ... ($y_1 + \cdots + y_h = 0$ for $h = 0$, etc.).

The conditions (3.3) must be satisfied since all the N-row numbers must be distinct.

Each system of values of a, b, ··· satisfying (3.3) when placed

in (3.2) gives a k-permutation. And each of the n cyclic permutations of each such k-permutation is also one (although all the associated i-permutations may not be distinct). We shall call these cyclic permutations <u>slides</u> of the given k-permutation. The totality of k-permutations (including their slides) will be called the set determined by the chain (and a particular starting point for the first link, as s_o above).

We prove the theorem:

<u>Theorem 3.1</u> <u>The two sets of k-permutations determined by starting the first link $(x_1 x_2 \cdots)$ at two different starting points are the same.</u>

Consider starting point s_o of P given by (3.2) and starting point s_o' of P' given by

(3.4) $$P' = \begin{bmatrix} s_o' & s_o' + x_1 & \cdots \\ x_1 & x_2 & \cdots & x_p \end{bmatrix} \begin{bmatrix} a' & a' + y_1 & \cdots \\ y_1 & y_2 & \cdots \end{bmatrix} \begin{bmatrix} b' & b' + z_1 & \cdots \\ z_1 & z_2 & \cdots \end{bmatrix} \cdots$$

If S indicates the operation of sliding a k-permutation by the amount of 1 step (to the right), then

(3.5) $$S^x P = \begin{bmatrix} s_o + x & s_o + x + x_1 & \cdots \\ x_1 & x_2 & \cdots \end{bmatrix} \begin{bmatrix} a + x & a + x + y_1 & \cdots \\ y_1 & y_2 & \cdots \end{bmatrix}$$
$$\begin{bmatrix} b + x & b + x + z_1 & \cdots \\ z_1 & z_2 & \cdots \end{bmatrix} \cdots$$

Now choose

(3.6) $$x = s_o' - s_o, \quad a' = a + (s_o' - s_o), \quad b' = b + (s_o' - s_o), \quad \cdots$$

Then we will have $S^x P = P'$ as is evident by inspection.

It must further be shown that the new parameters a', b', $\cdots$ defined in (3.6) satisfy conditions (3.3) with s_o' replacing s_o and

and a', b', $\cdots$ replacing a, b, $\cdots$.

From (3.3) we have

$$a + (s_o' - s_o) + \sum y \not\equiv s_o + (s_o' - s_o) + \sum x,$$

$$\text{or } a' + \sum y \not\equiv s_o' + \sum x, \text{ etc.}$$

(Note that the k-permutation zeros of P transform into those of P'. These zeros can be considered as links of length 1.)

There is thus no loss of generality in starting the first chain at value 0 of N.

<u>Theorem 3.2</u>. <u>Two t-chains are equivalent if the links of one are cyclic permutations of the other</u>.

The following proof for t = 3 is sufficient to illustrate the general situation.

Let

$$P = \begin{bmatrix} 0 & x_1 & x_1 + x_2 & \cdots \\ x_1 & x_2 & x_3 & \cdots \end{bmatrix} \begin{bmatrix} a & a + y_1 \cdots & a + y_1 + \cdots + y_{h-1} & \cdots \\ y_1 & y_2 \cdots & y_h & \cdots \end{bmatrix} \begin{bmatrix} b & b + z_1 & \cdots \\ z_1 & z_2 & \cdots \end{bmatrix}$$

$$P' = \begin{bmatrix} 0 & x_1 & x_1 + x_2 & \cdots \\ x_1 & x_2 & x_3 & \cdots \end{bmatrix} \begin{bmatrix} a' & a' + y_h & \cdots \\ y_h & y_{h+1} & \cdots \end{bmatrix} \begin{bmatrix} b' & b' + z_1 & \cdots \\ z_1 & z_2 & \cdots \end{bmatrix}$$

If now we choose $a' = a + (y_1 + \cdots + y_{h-1})$, $b' = b$, then P' reduces to P.

To show a', b' satisfy the (3.3) conditions we have, for example,

$$a + (y_1 + \cdots + y_{h-1}) + (y_h + y_{h+1} + \cdots) = a + (y_1 + y_2 + \cdots) \not\equiv x_1 + x_2 + \cdots,$$

$$b + (z_1 + z_2 + \cdots) \not\equiv a + y_1 + \cdots + y_h + y_{h+1} + \cdots,$$

or

$$a' + (y_h + y_{h+1} + \cdots) \not\equiv x_1 + x_2 + \cdots,$$
$$b' + (z_1 + z_2 + \cdots) \not\equiv a' + (y_h + y_{h+1} + \cdots).$$

As an illustration we have that chains (1, 2, 7)(3, 8, 9)(2, 8) and (7, 1, 2)(8, 9, 3)(8, 2) are equivalent.

<u>Theorem 3.3</u>. <u>Any permutation of the links of a t-chain gives an equivalent t-chain</u>.

It is sufficient to prove for an interchange of adjacent links.

Let

$$P = \begin{bmatrix} o & x_1 & \cdots \\ x_1 & x_2 & \cdots \end{bmatrix} \begin{bmatrix} a & a+y_1 & \cdots \\ y_1 & y_2 & \cdots \end{bmatrix} \begin{bmatrix} b & b+z_1 & \cdots \\ z_1 & z_2 & \cdots \end{bmatrix} \begin{bmatrix} c & c+u_1 & \cdots \\ u_1 & u_2 & \cdots \end{bmatrix},$$

$$P' = \begin{bmatrix} o & x_1 & \cdots \\ x_1 & x_2 & \cdots \end{bmatrix} \begin{bmatrix} a' & a'+y_1 & \cdots \\ y_1 & y_2 & \cdots \end{bmatrix} \begin{bmatrix} c' & c'+u_1 & \cdots \\ u_1 & u_2 & \cdots \end{bmatrix} \begin{bmatrix} b' & b'+z_1 & \cdots \\ z_1 & z_2 & \cdots \end{bmatrix}$$

We have

$$(3.7) \qquad \begin{cases} b + \Sigma z \not\equiv \Sigma x,\ a + \Sigma y, \\ c + \Sigma u \not\equiv \Sigma x,\ a + \Sigma y,\ b + \Sigma z, \end{cases}$$

$$(3.8) \qquad \begin{cases} c' + \Sigma u \not\equiv \Sigma x,\ a' + \Sigma y \\ b' + \Sigma z \not\equiv \Sigma x,\ a' + \Sigma y,\ c' + \Sigma u \end{cases}$$

Choose $a' = a$, $b' = b$, $c' = c$ and (3.8) reduces to (3.7),

<u>Theorem 3.4</u>. <u>Two chains are equivalent if at least one k-permutation produced by one is identical (to within a slide) to a k-permutation produced by the other</u>.

Let the chains be P, Q. Their links may be arranged so they start

$$P = (p, x_1, x_2, \cdots, x_g)(\cdots)\cdots,\ Q = (p, y_1, y_2, \cdots, y_h)(\cdots)\cdots,$$

or

$$P = \begin{bmatrix} 0 & p & p+x_1 & \cdots \\ p & x_1 & x_2 & \cdots \end{bmatrix} \cdots, \qquad Q = \begin{bmatrix} 0 & p & p+y_1 & \cdots \\ p & y_1 & y_2 & \cdots \end{bmatrix} \cdots$$

In this form the k-permutations in question will be identical without sliding. It follows that in this k-permutation

interval from p to x_1 = interval from p to y_1, $\therefore x_1 = y_1$,

interval from x_1 to x_2 = interval from y_1 to y_2, $\therefore x_2 = y_2$,

etc.

Now if $g < h$ we would get

$$0 \equiv p+x_1 + \cdots + x_g = (p+y_1 + \cdots + y_g) + (y_{g+1} + \cdots + y_h),$$

so that $y_{g+1} + \cdots + y_h \equiv 0$, a contradiction. If $g > h$ we get a like contradiction. Hence $g = h$, and the two first links of P and Q are the same.

A continuation of this argument will show that the links of P and Q are identical in pairs (to within cyclic permutations within links) and hence P and Q are equivalent.

This theorem shows that any two non-equivalent chains must always produce non-equivalent k-permutations. However as will be seen below a given chain can produce equivalent k-permutations.

4. Examples.

In this section we determine the coefficients of some terms so as to illustrate in some detail the two steps outlined at the end of section 2. Properties of links and chains proved in section 3 are used without special comment.

(1) Find the coefficient A of the term $a_0^3 a_1 a_2 a_3^3 a_6$ of C_9.

The non-zero subscripts are 1, 2, 3, 3, 3, 6. To find all their link-permutations we write down all $5!/3! = 20$ permutations keeping one element (say 1) fixed. These 20 are

123336	133236	136323
123363	133263	136332
123633	133326	162333
126333	133362	163233
132336	133623	163323
132363	133632	163332
132633	136233	

By inspection we obtain the seven 2-link chains

(1,2,3,3)(3,6)   (1,3,2,3)(6,3)   (1,3,3,2)(6,3)

(1,2,3,3)(6,3)   (1,3,3,2)(3,6)   (1,2,6)(3,3,3)

(1,6,2)(3,3,3)

There are no 1-link chains as is seen by direct test.

Of the seven 2-chains we select the five distinct (non-equivalent) 2-chains:

(a) (1,2,3,3)(3,6)   (b) (1,3,2,3)(3,6)   (c) (1,3,3,2)(3,6)

(d) (1,2,6)(3,3,3)   (e) (1,6,2)(3,3,3)

The cases (a), (b), (c) will be denoted by the single notation [1 2 3 3] [3 6] where the bracket indicates an <u>unordered sequence</u> all permutations of which will give links. Likewise cases (d) and (e) can be combined into [1 2 6] [3 3 3] .

We next find the k-permutations determined by each case.

Case (a) gives in a former notation

$$\begin{bmatrix} 0 & 1 & 3 & 6 \\ 1 & 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} a & a+3 \\ 3 & 6 \end{bmatrix}$$

and parameter

$$a \not\equiv 0,\ 1,\ 3,\ 6$$

$$a+3 \not\equiv 0,\ 1,\ 3,\ 6,$$

giving a = 2, 4, 5, 8. These give four k-permutations:

a \ N	0	1	2	3	4	5	6	7	8
2	1	2	$\underline{3}$	3	·	6	3	·	·
4	1	2	·	3	$\underline{3}$	·	3	6	·
5	1	2	·	3	·	$\underline{3}$	3	·	6
8	1	2	6	3	·	·	3	·	$\underline{3}$

The zeros are denoted by a dot. The underlined 3 is started at the value of a. Thus the k-permutation corresponding to a = 2 is 123306300, etc.

Case (b) gives parameter a = 2, 5, 8 and the three k-permutations

	0	1	2	3	4	5	6	7	8
2	1	3	$\underline{3}$	·	2	6	3	·	·
5	1	3	·	·	2	$\underline{3}$	3	·	6
8	1	3	6	·	2	·	3	·	$\underline{3}$

Case (c) gives a = 2, 3, 5, 8

	0	1	2	3	4	5	6	7	8
2	1	3	$\underline{3}$	·	3	6	·	2	·
3	1	3	·	$\underline{3}$	3	·	6	2	·
5	1	3	·	·	3	$\underline{3}$	·	2	6
8	1	3	6	·	3	·	·	2	$\underline{3}$

Case (d) is

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 2 & 6 \end{bmatrix} \quad \begin{bmatrix} a & a+3 & a+6 \\ 3 & 3 & 3 \end{bmatrix}$$

with $a, a+3, a+6 \not\equiv 0, 1, 3$. Hence a = 2, 5, 8.

We then obtain

	0	1	2	3	4	5	6	7	8
2	1	2	$\underline{3}$	6	·	3	·	·	3
5	1	2	3	6	·	$\underline{3}$	·	·	3
8	1	2	3	6	·	3	·	·	$\underline{3}$

Instead of 3 distinct k-permutations we here have only one.

Case (e) gives

	0	1	2	3	4	5	6	7	8
2	1	6	$\underline{3}$	·	·	3	·	2	3
5	1	6	3	·	·	$\underline{3}$	·	2	3
8	1	6	3	·	·	3	·	2	$\underline{3}$

Again there is only one distinct k-permutation.

We now count a total of $4+3+4+1+1=13$ distinct k-permutations produced by the five 2-chains, and each of these 13 can be slid through the complete set of 9 positions thus giving the value $(9)(13)=117$ as the value of A the coefficient. This term is what we would call a 2-chain term since that is the maximum number of links present.

(2) Find the coefficient A of the term $a_0^3 a_1 a_2^2 a_7^2 a_8$ in C_9.

There are 30 permutations of the non-zero subscripts 1, 2, 2, 7, 7, 8 (keeping element 1 fixed.) (If an element occurring more than once as 7 were kept fixed we would get **60** permutations but half of these would be slides of the other half and could be neglected. We can avoid this duplication by keeping fixed an element occurring only once if such is present.)

Of these 30 there are 10 link-permutations indicated by

(1,8)(2,7)(2,7)	(1,2,8,7)(2,7)	(1,2,2,8,7,7)
(1,8)(2,7)(7,2)	(1,2,8,7)(7,2)	(1,7,7,8,2,2)
(1,8)(7,2)(2,7)	(1,7,8,2)(7,2)	
(1,8)(7,2)(7,2)		

giving 5 non-equivalent link-permutations

(a) (1,2,2,8,7,7) (b) (1,7,7,8,2,2) (c) (1,2,8,7)(2,7)

(d) (1,7,8,2)(2,7) (e) (1,8)(2,7)(2,7)

Cases (a) and (b) are 1-chains giving the respective distinct k-permutations

	0	1	2	3	4	5	6	7	8
(a)	1	2	7	2	7	8	•	•	•
(b)	1	7	•	•	•	2	8	2	7

Cases (c) and (d) are 2-chains with respective parameter values $a = 4, 5, 6$ and $a = 2, 3, 4$. These give 6 distinct k-permutations:

(c)		0	1	2	3	4	5	6	7	8
	4	1	2	7	8	2	•	7	•	•
	5	1	2	7	8	•	2	•	7	•
	6	1	2	7	8	•	•	2	•	7

(d)		0	1	2	3	4	5	6	7	8
	2	1	7	2	•	7	•	•	2	8
	3	1	7	•	2	•	7	•	2	8
	4	1	7	•	•	2	•	7	2	8

Case (e) will be more involved since it contains two parameters. We have

$$\begin{bmatrix} 0 & 1 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} a & a+2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} b & b+2 \\ 2 & 7 \end{bmatrix}$$

with $a \not\equiv 0, 1, 7, 8$ and $b \not\equiv 0, 1, 7, 8;\ a,\ a+2,\ a+7$.

This gives 14 pairs of solutions (a, b):

	0	1	2	3	4	5	6	7	8
(2,3)	1	8	2	2	7	7	•	•	•
(2,5)	1	8	2	•	7	2	•	7	•
(2,6)	1	8	2	•	7	•	2	•	7
(3,2)	1	8	2	2	7	7	•	•	•
(3,4)	1	8	•	2	2	7	7	•	•
(3,6)	1	8	•	2	•	7	2	•	7
(4,3)	1	8	•	2	2	7	7	•	•
(4,5)	1	8	•	•	2	2	7	7	•
(5,2)	1	8	2	•	7	2	•	7	•
(5,4)	1	8	•	•	2	2	7	7	•
(5,6)	1	8	•	•	•	2	2	7	7
(6,2)	1	8	2	•	7	•	2	•	7
(6,3)	1	8	•	2	•	7	2	•	7
(6,5)	1	8	•	•	•	2	2	7	7

By inspection we see that (a,b) and (b,a) give identical k-permutations, thus reducing the number of distinct k-permutations for case (e) to 7.

There are thus a totalof $2 + 6 + 7 = 15$ distinct k-permutations each of which can be slid through all 9 positions. Hence the coeffi-

cient is (9)(15) = 135.

This term would be called a 3-chain term.

Note also that the two 2-link chain cases (c) and (d) are made up by combining two of the 3 links of the 3-link chain case (e).

We shall speak of the total (9)(2) = 18 determined by the 1-chains (cases (a), (b)) as the contribution of the 1-chains to the total coefficient, and the totals of (9)(6) = 54 (cases (c), (d)), and (9)(7) = 63 (case (e)) as the 2-chain and 3-chain contributions.

5. Special links and chains.

A. Cyclic 1-link chains. There are certain types of chains for which the associated set of k-permutations will contain duplicates. Some of the examples above show this property. We consider this property in this section.

Duplicate k-permutations will result if a given k-permutation has a cyclic character so that in sliding it through a cycle-interval we obtain the identical k-permutation again.

Consider first the 1-link chains with this property, and an example: in finding the coefficient of $a_0^4 a_7^4 a_8^4$ (n = 12), the 1-chains

(a) (7, 7, 8, 8, 7, 7, 8, 8), (b) 7, 8, 7, 8, 7, 8, 7, 8)
give respectively k-permutations

(a') 7 7 8 0 8 0 7 7 8 0 8 0, (b') 7 8 0 7 8 0 7 8 0 7 8 0

Here (a') consists of two cycles of length 6, and (b') consists of four cycles of length 3. Hence we could slide (a') through 6 positions only and (b') through 3 positions (lengths of the cycles).

Note also that the chains (a), (b) themselves are cyclic. We now show this situation always holds, i.e. a cyclic 1-chain always gives a cyclic k-permutation.

We take the general cyclic 1-chain

$$(5.1)\qquad P = (p_1, p_2, \cdots, p_m, p_1, p_2, \cdots, p_m, \cdots, p_1, p_2, \cdots, p_m)$$

containing c cycles of length m. Write P in the form

$$(5.2)\qquad P = \begin{bmatrix} 0 & p_1 & p_1+p_2 \cdots p_1+\cdots+p_{m-1} & s & s+p_1 \cdots (c-1)s+p_1+\cdots+p_{m-1} \\ p_1 & p_2 & p_3 \quad \cdots \quad p_m & p_1 & p_2 \quad \cdots \quad p_m \end{bmatrix}$$

where

$$(5.3)\qquad s \equiv p_1 + p_2 + \cdots + p_m$$

Hence $cs \equiv 0$ and c is the minimum value ($\neq 0$) satisfying this congruence.

From (5.2) it follows that

$$(5.4)\qquad S^s P = S^{2s} P = \cdots = S^{cs} P = P,$$

where

$$S^s P = \begin{bmatrix} s & s+p_1 & \cdots & 2s & 2s+p_1 & \cdots & 3s & \cdots \\ p_1 & p_2 & \cdots & p_1 & p_2 & \cdots & p_1 & \cdots \end{bmatrix}$$

or

$$(5.5)\qquad S^s K = S^{2s} K = \cdots = S^{cs} K = K,$$

where K denotes the k-permutation.

Now if we put $d = (s,n)$ it will follow that

$$(5.6)\qquad n = cd,$$

$$(5.7)\qquad c's \equiv d, \ (c' < c).$$

Hence from (5.5) and (5.7) we have

$$(5.8)\qquad S^d K = K,$$

so that K must have the form

$$(5.9)\qquad K = k_0 k_1 \cdots k_{d-1} k_0 k_1 \cdots k_{d-1} \cdots k_0 k_1 \cdots k_{d-1},$$

containing $c = n/d$ cycles of length d.

The chain P and therefore K contains exactly c of each p_i. Hence each cycle of K must contain $p_1, p_2, \cdots, p_m$ once each (remainder of k's in a cycle are zeros), i.e. each cycle of K consists of zeros and the cycle of P in some order (d-m zeros). A non-cyclic 1-chain corresponds to $c = 1$ or $d = n$, and K is non-cyclic also. We state the above results in the theorem:

Theorem 5.1. A cyclic 1-link chain P of c cycles produces a cyclic k-permutation K of c cycles each of length $d = n/c$. The cycle of K consists of the cycle of P and zeros in some order.

We can obtain a formula for the contribution A_1 to the coefficient A of any term due to the 1-link chains associated with the term:

Let P_c be the number of distinct 1-chains each containing c cycles with length of cycle equal d. Then dP_c will equal the number of k-permutations derived from these P_c 1-chains. Hence

$$A_1 = \sum dP_c = \sum_c nP_c/c$$

where the sum is over all divisors c of n.

Now let $K_c = k_0 k_1 \cdots k_{d-1}$ be the cycle of one of the k-permutations K, where k_0 is always used as the fixed (first) element of all the chains and suppose k_0 occurs n_c times in K_c. Then $m_1 = cn_c$, where m_1 is the total number of k_0's in any K (or 1-chain).

Hence

$$A_1 = \sum_c \frac{n}{m_1} (P_c n_c)$$

But $\sum_c n_c P_c$ is precisely the total number N of 1-link permutations beginning with k_0 (each set of $n_c P_c$ such permutations gives only P_c distinct 1-chains).

Then $N = \sum n_c P_c$ and finally

(5.10) $$A_1 = \frac{nN}{m_1}$$

N can be determined by counting the number of 1-link permutations all starting with k_o.

To illustrate the above reasoning we calculate A_1 for the term $a_o^4 a_7^4 a_8^4$ $(n = 12)$. The total set of 1-link permutations is given by

77788788	77878788	78778788	78778878
77887887	78787887	77878878	77887878
78878877	78788778	78788787	78878787
78877788	78877878	78878778	78787788
77887788	78787878		
78877887.			

Here $k_o = 7$; $c = 1, 2, 4$; $P_1 = 4$, $P_2 = 1$, $P_4 = 1$; $n_1 = 4$, $n_2 = 2$, $n_4 = 1$; $n_1P_1 = 16$, $n_2P_2 = 2$, $n_4P_4 = 1$; $m_1 = 4$.

Also

$$N = \sum n_c P_c = 16 + 2 + 1 = 19,$$

$\sum P_c = 4 + 1 + 1 = 6$ = number of distinct 1-chains,

and

$$A_1 = \frac{(12)(19)}{4} = 57$$

Note that A_1 is always an integer (but not necessarily a multiple of n.)

It is now easy to find the <u>complete coefficient</u> $A(= A_1)$ of any term which is a <u>1-chain term</u>. For let such a term be

(5.11) $$A a_o^{n-r} \; a_{p_1}^{m_1} \, a_{p_2}^{m_2} \cdots a_{p_h}^{m_h} \,, \; (\sum m_i = r),$$

where the p_i are distinct. To find N we form all permutations of

the set of $p_1, p_2, \cdots, p_h$ keeping p_1 fixed (in first position say).

Then

$$N = \frac{(r-1)!}{(m_1-1)!\, m_2! \cdots m_h!}$$

Hence by (5.10) we have

$$\boxed{A = A_1 = \frac{n}{m_1} N = \frac{n(r-1)!}{m_1!\, m_2! \cdots m_h!}} \tag{5.12}$$

For example the 1-chain term

$Aa_o^4 a_1^4 a_2^4$ of a C_{12} has

$$A = \frac{(12)(7!)}{4!\ 4!} = 105$$

That this is a 1-chain term follows since its (non-zero) subscripts 1, 1, 1, 1, 2, 2, 2, 2, have a sum of 12. It will be recalled that any term as (5.11) is said to be a 1-chain term <u>if every subset of its r (non-zero) subscripts has a sum incongruent to zero (mod n)</u>. In particular any term of weight n is a 1-chain term (as $a_o^4 a_1^4 a_2^4$ above) but of course terms of weight greater than n can be such also.

B. <u>Cyclic 2-link chains</u>.

The following discussion will be treated in a simplified and more general form in the next section but it is given here as it will be of value in a later problem.

Consider then a general 2-link chain PQ given by (assume P not identical to Q)

$PQ = (p_1, p_2, \cdots, p_m, \cdots, p_1, p_2, \cdots, p_m)(q_1, q_2, \cdots, q_j, \cdots, q_1, q_2, \cdots, q_j)$ containing c cycles in P and c' cycles in Q. Put $s \equiv p_1 + \cdots + p_m$, $s' \equiv q_1 + \cdots + q_j$, so $cs \equiv 0$, $c's' \equiv 0$ and c, c'

are the minimum (non-zero) values satisfying these respective congruences. Put also

$$X_a = \begin{bmatrix} 0 & p_1 & \cdots & s & \cdots & 2s & \cdots \\ p_1 & p_2 & \cdots & p_1 & \cdots & p_1 & \cdots \end{bmatrix} \begin{bmatrix} a & a+q_1 & \cdots & a+s' & \cdots & a+2s' & \cdots \\ q_1 & q_2 & \cdots & q_1 & \cdots & q_1 & \cdots \end{bmatrix} = PQ_a$$

Then $(S^s, S^{2s}, \cdots, S^{cs})\, P = P$, $(S^{s'}, S^{2s'}, \cdots, S^{c's'})Q_a = Q_a$.

If $d = (s, n)$, $d' = (s', n)$ we will have $n = cd = c'd'$ and also that

$$S^d P = P, \; S^{d'} Q_a = Q_a$$

If $e =$ l.c.m. (d, d') then

$$S^e X_a = S^e(PQ_a) = PQ_a = X_a$$

Hence

(5.13) $$S^e K_{(a)} = K_{(a)}$$

where $K_{(a)}$ is the k-permutation corresponding to the parameter value a. From (5.13) it follows that a slide of e duplicates $K_{(a)}$, so we can write

$$K_{(a)} = k_o k_1 \cdots k_{e-1} k_o \cdots k_{e-1} \cdots k_o \cdots k_{e-1}$$

If we put $h = (c, c')$ then we will have $n = he$, so every $K_{(a)}$ consists of h cycles of length e.

Now

$$S^{us}Q_a = \begin{bmatrix} a+us & \cdots & a+s'+us & \cdots \\ q_1 & \cdots & q_1 & \cdots \end{bmatrix} = \begin{bmatrix} a' & \cdots & a'+s' & \cdots & a'+2s' & \cdots \\ q_1 & \cdots & q_1 & \cdots & q_1 & \cdots \end{bmatrix} = Q_{a'}$$

where $a' = a+us$, and it can be shown that a' is an allowable parameter value if a is. But $us \equiv d$ for some u, so that

$$S^d Q_a = Q_{a'} = Q_{a+d},$$

or

(5.14) $$S^d K_{(a)} = K_{(a+d)},$$

i.e. a slide of d on $K_{(a)}$ produces $K_{(a+d)}$. (Note that (5.14) and also (5.13) hold for all allowable values of parameter a.)

It follows that the allowable a-values split into sets

$$
\begin{array}{l}
a_1,\ a_1+d,\ a_1+2d,\ \cdots,\ a_1+(c-1)d \\
a_2,\ a_2+d,\ a_2+2d,\ \cdots,\ a_2+(c-1)d \\
\vdots \qquad\qquad\qquad\qquad \vdots \\
a_x,\ a_x+d,\ \cdots\cdots,\ a_x+(c-1)d
\end{array}
$$

such that all $K_{(a_i+ud)}$ are equivalent (by sliding) ($i = 1, 2, \cdots, x$; $u = 0, 1, \cdots, c-1$).

If then N_a be the total number of allowable a-values, i.e. such that $a + (q_1 + \cdots + q_i) \not\equiv (p_1 + \cdots + p_t) + ud$,
$\left(\begin{array}{l} q_0 = p_0 = 0;\ i = 1, \cdots, m-1; \\ t = 1, \cdots, j-1;\ u = 0, 1, \cdots, c-1 \end{array}\right)$, we will have

$$N_a = cx \tag{5.16}$$

We know $Q_{a+vd'} = Q_a$, and it can be shown $a+vd'$ is an allowable parameter value with a. Hence $X_{a+vd'} = X_a$, ($v = o, 1, \cdots, c'-1$), or

$$K_{(a+vd')} = K_{(a)} \tag{5.17}$$

Hence again the a-values can be split into sets

$$
\begin{array}{l}
a'_1,\ a'_1+d',\ a'_1+2d',\ \cdots,\ a'_1+(c'-1)d' \\
\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
a'_y,\ a'_y+d',\ a'_y+2d',\ \cdots,\ a'_y+(c'-1)d'
\end{array}
\tag{5.18}
$$

so that

$$N_a = c'y = cx \tag{5.19}$$

Consider next $S^d K_{(a'_t)}$. From (5.14) we have

$$S^d K_{(a'_t)} = K_{(a'_t+d)}$$

Now a'_t+d must occur in one of the rows of (5.18):

(1) if it occurs in the a'_t row,

$$a'_t + d \equiv a'_t + vd', \text{ and}$$

(5.20) $$S^d K_{(a'_t)} = K_{(a'_t+d)} = K_{(a'_t+vd')} = K_{(a'_t)},$$

so a slide of d on $K_{(a'_t)}$ reproduces it.

(2) if $a'_t + d$ occurs in the a'_w row $(w \neq t)$,

(5.21) $$S^d K_{(a'_t)} = K_{(a'_t+d)} = K_{(a'_w+vd')} = K_{(a'_w)}$$

It is then seen from (5.20), (5.21) that every $K_{(a'_t)}$, (i.e. every $K_{(a)}$) can be slid only d distinct steps. We have then that the contribution to the coefficient of the term due to the 2-link chain PQ is

(5.22) $$dy = d\frac{N_a}{c'} = \frac{nN_a}{cc'}$$

Finally we have the situation in which P and Q are identical. For this case the formula (5.22) will be changed by a factor. Now $s = s'$, $d = d' = e$, $c = c'$, and $X_a = PP_a$.

It can be shown that if a is an allowable value then so also is $\bar{a}$ where $a + \bar{a} = n$. We will have $S^{\bar{a}}X_a = X_{\bar{a}}$ or

(5.23) $$S^{\bar{a}}K_{(a)} = K_{(\bar{a})}$$

As before $N_a = cx$, and there are $y = x$ values of a: $a_1, a_2, \cdots, a_x$ whose k-permutations are not identical (see 5.18).

Suppose there are x_1 values of $a_1, \cdots, a_x$ such that $a_i + ud \not\equiv \bar{a}_i$ (see 5.15). These x_1 values occur in complementary pairs $(a_i, \bar{a}_i)$ and for each of these there will be d distinct slides of $K_{(a_i)}$. Hence this set of x_1 values contribute the amount $\underline{(d)(\frac{x_1}{2})}$ to the coefficient.

Let $x_2(=x-x_1)$ be the remainder of the x a's for which $a_i + ud \equiv \bar{a}_i$.

Hence $2a_i + ud \equiv 0$, and since $a_i \not\equiv vd$ (by its definition) we must have $a_i \equiv (n-u)\frac{d}{2}$, and n-u must be odd; so we can write $a_i \equiv td + \frac{d}{2}$.

Also

$$S^{\bar{a}_i}K_{(a_i)} = K_{(\bar{a}_i)} = K_{(a_i+ud)} = K_{(a_i)} = S^{a_i+ud}K_{(a_i)} = S^{a_i}K_{(a_i)}$$

Hence, $S^{td+d/2}K_{(a_i)} = S^{\frac{d}{2}}K_{(a_i)} = K_{(a_i)}$, i.e. there are only $\frac{d}{2}$ distinct slides possible for these latter x_2 cases. Hence we obtain a contribution of $\frac{d}{2}x_2$ to the coefficient.

We finally have a total of

$$d\left(\frac{x_1}{2}\right)+\left(\frac{d}{2}\right)x_2 = \tfrac{1}{2}dx = \tfrac{1}{2}\frac{nN_a}{c^2}$$

as the complete contribution. Note that this is $\frac{1}{2}$ the previous amount, (5.22).

In the cases above considered it is possible to obtain some information about the size of N_a. We state without proof the following:

$$N_a = n - z\frac{cc'}{h}$$

where z in an integer such that $1 \leq z \leq mj$. In particular if $n = cc'/h$ we must have $z = 1$ and $N_a = 0$.

In the case where P is identical to Q we have

$$N_a = n - zc, \qquad m \leq z \leq m^2 - m + 1.$$

The precise value of z will depend on the relations between the p's and q's, the elements of the two links. We determine the exact value of N_a for a variety of situations in later sections.

6. A general Formula for coefficients.

In this section we give a formula for the coefficient of any term, this formula being derived from the types of links and chains associated with the term. First we find a formula for the contribution to the coefficient of any particular t-link chain. We illustrate the general proof by means of a 4-link chain:

$$(6.1)\quad \begin{aligned}&(p_1p_2\cdots p_m\cdots p_1\cdots p_m)(p_1p_2\cdots p_m\cdots p_1\cdots p_m)\\&(q_1q_2\cdots q_j\cdots q_1\cdots q_j)(q_1q_2\cdots q_j\cdots q_1\cdots q_j) = PPQQ\end{aligned}$$

where the first two chains P, P are identical with c_1 cycles of length m, and the second two Q, Q are identical with c_2 cycles of length j.

To gain the advantage of symmetry we start the first chain at a general position a_1 instead of at 0 as heretofore. We write then (6.1) as

$$(6.2)\quad \begin{bmatrix}a_1 & a_1+p_1 & \cdots & a_1+s_1 & \cdots & a_1+2s_1 & \cdots\\ p_1 & p_2 & \cdots & p_1 & \cdots & p_1 & \cdots\end{bmatrix}\begin{bmatrix}a_2 & \cdots & a_2+s_1 & \cdots\\ p_1 & \cdots & p_1 & \cdots\end{bmatrix}\begin{bmatrix}a_3 & \cdots & a_3+s_2 & \cdots\\ q_1 & \cdots & q_1 & \cdots\end{bmatrix}$$
$$\begin{bmatrix}a_4 & \cdots & a_4+s_2 & \cdots\\ q_1 & \cdots & q_1 & \cdots\end{bmatrix} = P_{a_1}P_{a_2}Q_{a_3}Q_{a_4}$$

where $s_1 = p_1 + \cdots + p_m, \ldots \quad s_2 = q_1 + \cdots + q_j$

Let $M(a_1, a_2, a_3, a_4)$ equal the total number allowable parameter-values of (a_1, a_2, a_3, a_4), and let $K(a_1, a_2, a_3, a_4)$ be the k-permutation derived from the parameter-values (a_1, a_2, a_3, a_4). We will then have the relations

$$(6.3)\quad K(a_1, a_2, a_3, a_4) = K(a_2, a_1, a_3, a_4) = K(a_1, a_2, a_4, a_3),$$

(6.4) $K(a_1, a_2, a_3, a_4) = K(a_1+u_1d_1, a_2+u_2d_1, a_3+v_1d_2, a_4+v_2d_2)$,

(6.5) $S^hK(a_1, a_2, a_3, a_4) = K(a_1+h, a_2+h, a_3+h, a_4+h)$.

In (6.4), $d_i = (s_i, n)$, $n = c_id_i$, $u_i = 0, 1, \cdots, c_1-1$, $v_i = 0, 1, \cdots, c_1-1$, and relations (6.5) hold for all h. We then have for the contribution C due to the particular chain (6.1)

(6.6) $$C = \left(\frac{1}{2!2!}\right)\left(\frac{1}{c_1^2\ c_2^2}\right) M$$

The first factor $\left(\frac{1}{2!\ 2!}\right)$ in C is due to the relations (6.3), and the second factor $\left(\frac{1}{c_1^2\ c_2^2}\right)$ is due to the relations (6.4).

Now we have

$$M(a_1, a_2, a_3, a_4) = nM(0, a_2, a_3, a_4).$$

Put $M(0, a, b, c) = N_{abc}$ so that N_{abc} gives the total number of parameter values (a, b, c) when the first chain is started at 0. We rewrite (6.6) then as

$$C = \frac{n}{2!\ 2!\ c_1^2\ c_2^2} N_{a_1a_2a_3}$$

(changing the notation to start parameter a_1 with the second link).

Coming now to the general t-link chain $P_1P_2\cdots P_t$ consisting of n_1 identical links of c_1 cycles each, n_2 identical links of c_2 cycles each, $\cdots$, n_r identical links of c_r cycles each we will obtain for its contribution C,

(6.7) $$C = \frac{n}{n_1!\ n_2!\ \cdots\ n_r!\ c_1^{n_1}\ c_2^{n_2}\ \cdots\ c_r^{n_r}} N_{a_1a_2\cdots a_{t-1}}$$

$$(t = n_1 + \cdots + n_r)$$

The sum of all these contributions (6.7) due to all possible (distinct) t-link chains $X_t = P_1P_2\cdots P_t$ is called A_t where

$$A_t = \sum \frac{n \; N_{a_1 \cdots a_{t-1}}(X_t)}{n_1! \cdots n_r! \; c_1^{n_1} \cdots c_r^{n_r}}, \tag{6.8}$$

and $N_{a_1 \cdots a_{t-1}}(X_t)$ is the value of $N_{a_1} \cdots a_{t-1}$ corresponding to a particular t-chain X_t, and the sum is over all such X_t.

To obtain the actual coefficient we must sum over all A_t values, i.e.

$$A = A_1 + A_2 + A_3 + \cdots + A_m \tag{6.9}$$

where the highest order chain associated with the term is an m-link chain.

To illustrate the formula consider the term $a_0^{12}a_3^2a_6^2a_9^2$, $(n = 18)$. There are no 1-link chains, so $A_1 = 0$, and no 3-link chains or higher (since the term's weight is 2(18), $m \leq 2$). For the 2-link chains we have

	N_{a_1}	C
(3,6,3,6,)(9,9)	14	$\frac{14}{(2)(2)}$
(3,3,6,6)(9,9)	12	$\frac{12}{2}$
(3,6,9)(3,6,9)	12	$\frac{12}{2!}$
(3,9,6)(3,6,9)	13	13
(3,9,6)(3,9,6)	12	$\frac{12}{2!}$

so that

$$A = A_2 = 18\left[\frac{14}{(2)(2)} + \frac{12}{2} + \frac{12}{2!} + 13 + \frac{12}{2!}\right] = 621$$

For values of $m > 4$ the calculations in determining the A_t may become very involved. In the following sections we give some methods to reduce these calculations. The use of a machine

for most of the calculations would still remain a necessity however. We also give later the various operations of such a machine.

7. Some Coefficients for general n. In Part I the coefficients of all terms of the form $a_o^{n-4}a_{p_1}a_{p_2}a_{p_3}a_{p_4}$ were found (p_i may be equal). These results have been extended to cover the $a_o^{n-5}a_{p_1}\cdots a_{p_5}$ and $a_o^{n-6}a_{p_1}\cdots a_{p_6}$ cases (see Summary). We give here the analysis for one of the a_o^{n-6} cases $a_o^{n-6}a_{p_1}a_{p_2}a_{p_3}^2a_{p_4}^2$. The methods used are general and will apply to any term of a_o^{n-r} type though for larger r the number of possibilities increases very rapidly.

Consider the term

$$a_o^{n-6}a_{p_1}a_{p_2}a_{p_3}^2a_{p_4}^2, \ (p_1, p_2, p_3, p_4 \neq).$$

The weight condition is

(7.1) $$p_1 + p_2 + 2p_3 + 2p_4 \equiv 0 \pmod n.$$

The first step is to classify all possible types of chains obtainable from the six subscripts $p_1, p_2, p_3, p_3, p_4, p_4$. A 3-link chain is the maximum since a link contains at least 2 elements.

Use (7.1) and decompose it in all possible ways to give 2 sets or 3 sets of sums each congruent to zero. Each such sum is a link. For example, we may have

(7.2) $$p_1 + p_2 \equiv 0, \qquad 2p_3 + 2p_4 \equiv 0$$

giving the 2-link chains $(p_1, p_2)(p_3, p_3, p_4, p_4)$, (p_1, p_2) (p_3, p_4, p_3, p_4). (here we are assuming $p_3 + p_4 \not\equiv 0$). We would indicate both these chains by the notation $[12][3344]$ where the subscripts on the p_i are used instead of the p_i themselves. A

bracket expression as [3344] is regarded as an <u>unordered link</u>.

Another possible decomposition of (7.1) is $p_1 + p_3 \equiv 0$, $p_2 + p_3 + p_4 + p_4 \equiv 0$ or [13] [2344] . Note here that [23] [1344] or [14] [2334] would be regarded as not distinct from [13] [2344] .

We find there are 11 distinct chain-types of the above nature obtainable from (7.1). These are

[3344] [12]   {[3344] [12] , [133] [244]}
[2344] [13]   {[2344] [13] , [124] [334]}
[1244] [33]   {[2344] [13] , [144] [233]}
[123] [344]   {[1244] [33] , [123] [344]}
[133] [244]   {[133] [244] , [12] [34] [34]}
[12] [34] [34]

Of course, in addition, we have the 1-chain type [123344] .

A type in the { } as {[3344] [12] , [133] [244]} means both

$$2p_3 + 2p_4 \equiv 0,\ p_1 + p_2 \equiv 0$$

and

$$p_1 + 2p_3 \equiv 0,\ p_2 + 2p_4 \equiv 0$$

Consider next, for example, the type [3344] [12] . We must find all the <u>link-permutations</u> associated with it. Write out all the 5!/2!2! = 30 permutations of 1, 2, 3, 3, 4, 4 (keeping element 1 fixed):

123344	2	132344	1	134234	1	142334	1	143342	X
123434	2	132434	1	134243	1	142343	1	143423	1
123443	2	132443	1	134324	1	142433	1	143432	X
124334	2	133244	1	134342	X	143234	1	144233	1
124343	2	133424	1	134423	1	143243	1	144323	1
124433	2	133442	X	134432	X	143324	1	144332	X

The number at the side of each permutation indicates the number of links it decomposes into. An X indicates no chain is possible.

Of the six 2-link chains, two are distinct:

(a) (3, 3, 4, 4)(1, 2)

(b) $(3, 4, 3, 4)(1, 2)$,

and there are 18 distinct 1-link chains.

For case (a) we have

$$\begin{bmatrix} 0 & p_3 & 2p_3 & 2p_3+p_4 \\ p_3 & p_3 & p_4 & p_4 \end{bmatrix} \begin{bmatrix} a & a+p_1 \\ p_1 & p_2 \end{bmatrix}$$

with $a \not\equiv 0, p_3, 2p_3, 2p_3+p_4$; $a+p_1 \not\equiv 0, p_3, 2p_3, 2p_3+p_4$, or

$a \not\equiv 0, p_3, 2p_3, 2p_3+p_4$; $p_2, p_2+p_3, p_2+2p_3, p_2+2p_3+p_4$.

(Note $-p_1 \equiv p_2$). <u>And these 8 values are incongruent to each other,</u> since any contrary assumption like $p_1+2p_3+p_4 \equiv p_3$ leads to $p_1+p_3+p_4 \equiv 0$, a contradiction since $\overline{3344}\ \overline{12}$ is assumed here as the only decomposition of our 6 p_i. It follows then that $N_a = n-8$. For case (b) we also find $N_a = n-8$. The value of A_2 the 2-link chains contribution is then

$$A_2(n) = n\left[(n-8)+\tfrac{1}{2}(n-8)\right] = \tfrac{3}{2}\, n\,(n-8)$$

(The $\frac{1}{2}$ is due to the cycle in the first link of case (b)).

The 18 1-link chains contribute

$$A_1(n) = 18n$$

Hence

$$A = A_1 + A_2 = \tfrac{3}{2}n^2 + 3n$$

is the coefficient of the term $a_o^{n-6}a_{p_1}a_{p_2}a_{p_3}^2a_{p_4}^2$ when the subscripts p_i are of type $\overline{3344}\ \overline{12}$.

An example of such a term is $a_o^4a_1a_9a_2^2a_3^2$ $(n = 10)$ with $A = 180$.

A similar analysis is carried out for each of the other types.

8. <u>Some special methods</u>.

A study of the example worked out in the previous section will show that no use was made of the particular numerical values of the

subscripts p_i in obtaining A_1, A_2, and A which are polynomials in n. It can be shown in fact that this is generally true, i.e. the functions $A_t(n)$ arising from a t-chain will be polynomials in n of degree t with coefficients whose values depend only on the chain-type and not on the particular subscript numerical values. The same must then be true of A the complete coefficient. For example consider the term $a_o^4a_1a_9a_2^2a_3^2$ considered above, and $a_o^4a_3a_7a_1^2a_4^2$ have both the same chain-type $\boxed{3344}$ $\boxed{12}$. Hence they must both have the same coefficient, 180.

The above facts can be used to advantage in the more difficult cases of chains of three or more links. For example, to find $A_3(n)$ for the term $a_o^{n-6}a_{p_1}a_{p_2}a_{p_3}a_{p_4}a_{p_5}a_{p_6}$ if the p_i (all unequal) are of the chain-type $\boxed{p_1p_2}$ $\boxed{p_3p_4}$ $\boxed{p_5p_6}$ (or $\boxed{12}$ $\boxed{34}$ $\boxed{56}$).

We use a specific set of p_i's of this type:

$$(1,\ n-1)(2,\ n-2)(4,\ n-4).$$

We have

$$X_{ab} = \begin{bmatrix} 0 & 1 \\ 1 & n-1 \end{bmatrix} \begin{bmatrix} a & a+2 \\ 2 & n-2 \end{bmatrix} \begin{bmatrix} b & b+4 \\ 4 & n-4 \end{bmatrix}$$

with $a \neq 0, 1, n-2, n-1$; $b \neq 0, 1, n-4, n-3; a, a+2, a-4, a-2$. Hence if $a = 2, 3, 4, 5, n-3, n-4, n-5, n-6$, b assumes $(n-7)$ values. For the remaining $(n-12)$ allowable a-values b takes on $(n-8)$ values. Hence

$$A_3(n) = \left[(n-12)(n-8) + 8(n-7)\right]n = n^3 - 12n^2 + 40n$$

A second method is to use several specific values of n, evaluate $A_t(n)$ directly for these values and then find the coefficients in the general $A_t(n)$ by the method of undetermined coefficients.

For example, to find $A_4(n)$ for the term $a_o^{n-8}a_{p_1}a_{p_2}\cdots a_{p_8}$ (all $p_i \neq$)

when its chain-type is $[p_1p_2]\ [p_3p_4]\ [p_5p_6]\ [p_7p_8]$. Assume $A_4(n) =$

$f_0n^4 + f_1n^3 + f_2n^2 + f_3n$ (there can be no constant term in $A_t(n)$),

and take $n = 17, 18, 19, 20$. For $n = 17$ we can use the chain

(1, 16)(2, 15)(4, 13)(8, 9)

From

$$\begin{bmatrix} 0 & 1 \\ 1 & 16 \end{bmatrix} \begin{bmatrix} a & a+2 \\ 2 & 15 \end{bmatrix} \begin{bmatrix} b & b+4 \\ 4 & 13 \end{bmatrix} \begin{bmatrix} c & c+8 \\ 8 & 9 \end{bmatrix}$$

we have

$$a \not\equiv 0, 1, 15, 16;\ b \not\equiv 0, 1, 13, 14;\ a,\ a+2,\ a-4,\ a-2;$$

(8.1)

$$c \not\equiv 0, 1, 9, 10;\ a,\ a+2,\ a-8,\ a-6;\ b,\ b+4,\ b-8,\ b-4$$

From the form of the 4-chain and (6.8) we see that we can write

$A_4(n) = nN^{(n)}_{abc} = n(n^3 + f_1n^2 + f_2n + f_3)$ where $N^{(n)}_{abc}$ is the number of

solutions (a, b, c) of (8.1). (Also the f_i are integers. Three values of n are now sufficient)

The following is a convenient method to evaluate $N^{(n)}_{abc}$. (We use n = 17). Construct three parameter-value Tables: T_{ab}, T_{ac}, T_{bc}. These are rectangular Tables. In the T_{ab} Table list vertically along the side all allowable values of parameter a (these are the integers from 0 through 16 excluding 0, 1, 15, 16 by (8.1)). Along the top of T_{ab} list all 17 integers 0 through 16. These are the b-values. For any particular value of a say a_0 cross out in the a_0 row of T_{ab} all values of b not allowed as given by (8.1). If $a_0 = 2$ we cross out 0, 1, 13, 14, 2, 4, 15, etc.

To construct T_{ac} we proceed in a similar manner except we disregard the b-values b, b + 4, b - 8, b - 4 in obtaining the excluded

c-values for a given a-value. So for $a_o = 2$ we would cross out for c: 0, 1, 9, 10, 2, 4, 11, 13.

To construct the T_{bc} Table we disregard the a, a + 2, a - 4, a - 2 in listing the possible b-values (we would merely exclude b = 0, 1, 13, 14) and disregard the a, a + 2, a - 8, a - 6 in listing possible c-values.

Portions of these Tables are given below:

T_{ab}		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 = b
	2	X	X	X	5	X	7	7	6	7	8	7	6	6	X	X	X	5
	3	X	X		X		X								X	X		X
a:	4	X	X	X		X		X							X	X		
	5	X	X		X		X		X						X	X		

T_{ac}		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 = c
	2	X	X	X		X					X	X	X		X			
	3	X	X		X		X				X	X		X		X		
a:	4	X	X			X		X			X	X			X		X	
	5	X	X				X		X		X	X				X		X

T_{bc}		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 = c
	2	X	X	X				X			X	X	X				X	
	3	X	X		X				X		X	X		X				X
b:	4	X	X			X				X	X	X			X			
	5	X	X				X				X	X				X		

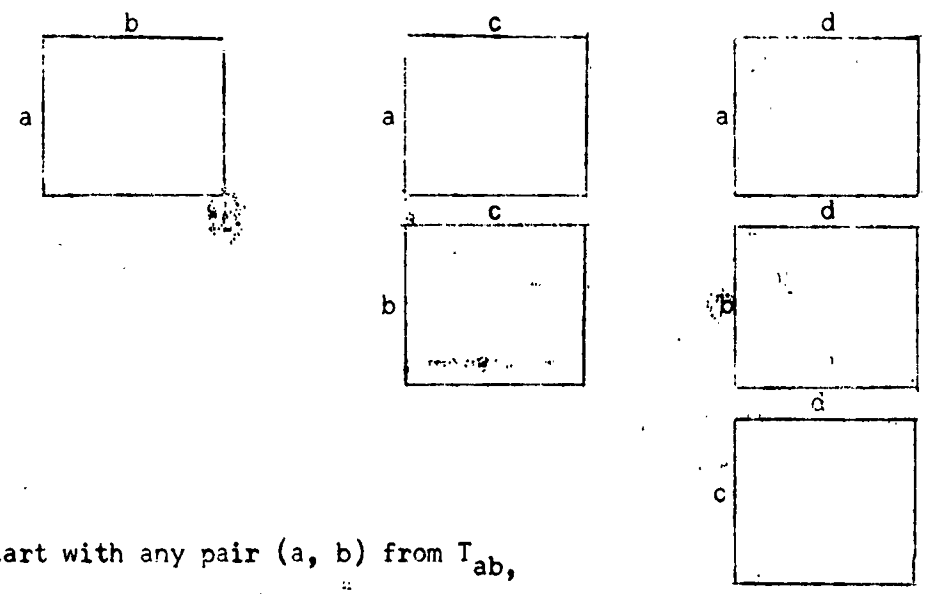
Now corresponding to a given allowable (a, b) pair, say (2, 3), found in T_{ab}, find in T_{ac} the a = 2 row and in T_{bc} the b = 3 row. The common values of c in these two rows give all allowable (2, 3, c) triples. Here they would be (2, 3, 5),

(2, 3, 6), (2, 3, 8), (2, 3, 14), (2, 3, 15), a total of 5 which can be recorded conveniently in the (2, 3) cell of T_{ab} (the first row of T_{ab} is shown thus recorded).

Continuing in this way we obtain a total of 841 for the number of (a, b, c) allowable-values, i.e. $N_{abc}(17) = 841 = 17^3 + f_1 \cdot 17^2 + f_2 \cdot 17 + f_3$.

Next we find $N_{abc}(18)$, $N_{abc}(19)$ from which we solve for f_1, f_2, f_3, which then determine $A_4(n)$.

The above procedure can be extended to calculate any $A_t(n)$. If $t = 5$ we would need to find $N^{(n)}_{abcd}$ for various values of n and for each chain which contributes to the $A_5(n)$. (In the above example there was only one chain so contributing). We would construct Tables:



(1) Start with any pair (a, b) from T_{ab},

(2) Find all (a, b, c) using T_{ac}, T_{bc},

(3) For each (a, b, c) find all (a, b, c, d) by T_{ad}, T_{bd}, T_{cd}.

By hand this process would become impractical for large t and n. However it is clear how a suitable machine can carry out the procedure which is essentially one of counting.

9. <u>The use of Ore's formula</u>. We have seen that the coefficient A of any term of a circulant permanent is expressible as a sum

(6.9) $$A = A_1 + A_2 + \cdots + A_m$$

where A_t is the contribution due to the t-chains. In this section we show how A can be calculated by evaluating only half of these A_i's, specifically the set $(A_1 + A_3 + A_5 + \cdots)$ or the set $(A_2 + A_4 + A_6 + \cdots)$.

This will depend on the formula given by Ore (Some studies on cyclic determinants, O. Ore, Duke Mathematical Journal v. 18, 1951, pp. 343-354) for the expansion of a circulant <u>determinant</u>. We give here a description of the use of his formula.

Suppose then we wish to evaluate the coefficient B of a term

(9.1) $$a_o^{n-r} a_{p_1} a_{p_2} \cdots a_{p_r}$$

of a circulant determinant (B will always be used to designate such a coefficient). The subscripts p_i may take equal values. Put

(9.2) $$P = p_1 + p_2 + \cdots + p_r \equiv 0.$$

Now form all partitions of r: $[r_1, r_2, \cdots, r_g]$ where $r_i > 1$, so

$g = 1$: $[r]$

$g = 2$: $[r - 2, 2]$, $[r - 3, 3]$, $[r - 4, 4]$, $\cdots$

$g = 3$: $[r - 4, 2, 2]$, $[r - 5, 2, 3]$, $\cdots$, etc.

For each such partition divide the sum P into g parts

$$P = P_1 + P_2 + \cdots + P_g,$$

where

$$\begin{aligned} P_1 &= p_1 + p_2 + \cdots + p_{x_1} && = \text{length } x_1, \\ P_2 &= p_{x_1+1} + \cdots + p_{x_1 + x_2} && = \text{length } x_2, \\ &\vdots && \vdots \\ P_g &= \cdots + p_r && = \text{length } x_g, \end{aligned}$$

and each $P_i \equiv 0 \pmod{n}$. In forming the P_i consider all the p's as formally different, and P is to be thus partitioned as $\sum P_i$ in all possible ways.

If now the term (9.1) be written in the equivalent form

$$a_0^{n-r} a_1^{m_1} a_2^{m_2} \cdots a_{n-1}^{m_{n-1}} ,$$

then

$$B = \frac{1}{m_1!\, m_2! \cdots m_{n-1}!} \sum (-1)^{r-g} (x_1-1)!(x_2-1)!\cdots(x_g-1)!\, n^g, \tag{9.3}$$

where the sum covers all partitions of P as above described.

An example will make the procedure clear: To find B of the term $Ba_0^4 a_2^2 a_6^2 a_8^4$ $(n=12)$.

Here $p_1 = p_2 = 2$, $p_3 = p_4 = 6$, $p_5 = p_6 = p_7 = p_8 = 4$, $r = 8$

We then have

$g=1$: $[8]$: $(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8)$, $x_1 = 8$

$g=2$: $[6, 2]$: $(p_1 + p_2 + p_5 + p_6 + p_7 + p_8) + (p_3 + p_4)$, $x_1 = 6$, $x_2 = 2$

$[5, 3]$: $(p_3 + p_4 + p_6 + p_7 + p_8) + (p_1 + p_2 + p_5)$, $x_1 = 5$, $x_2 = 3$ for all $[5,3]$

$(p_3 + p_4 + p_5 + p_7 + p_8) + (p_1 + p_2 + p_6)$

$(p_3 + p_4 + p_5 + p_6 + p_8) + (p_1 + p_2 + p_7)$

$(p_3 + p_4 + p_5 + p_6 + p_7) + (p_1 + p_2 + p_8)$

$(p_1 + p_2 + p_3 + p_4 + p_8) + (p_5 + p_6 + p_7)$

$(p_1 + p_2 + p_3 + p_4 + p_7) + (p_5 + p_6 + p_8)$

$(p_1 + p_2 + p_3 + p_4 + p_6) + (p_5 + p_7 + p_8)$

$(p_1 + p_2 + p_3 + p_4 + p_5) + (p_6 + p_7 + p_8)$

$[4, 4]$: $(p_1 + p_3 + p_5 + p_6) + (p_2 + p_4 + p_7 + p_8)$, $x_1 = 4$, $x_2 = 4$

$(p_1 + p_3 + p_5 + p_7) + (p_2 + p_4 + p_6 + p_8)$ for all $[4, 4]$

$(p_1 + p_3 + p_5 + p_8) + (p_2 + p_4 + p_6 + p_7)$

$(p_1 + p_3 + p_6 + p_7) + (p_2 + p_4 + p_5 + p_8)$

$(p_1 + p_3 + p_6 + p_8) + (p_2 + p_4 + p_5 + p_7)$

$(p_1 + p_3 + p_7 + p_8) + (p_2 + p_4 + p_5 + p_6)$

$(p_1 + p_4 + p_5 + p_6) + (p_2 + p_3 + p_7 + p_8)$

$(p_1 + p_4 + p_5 + p_7) + (p_2 + p_3 + p_6 + p_8)$

$(p_1 + p_4 + p_5 + p_8) + (p_2 + p_3 + p_6 + p_7)$

$(p_1 + p_4 + p_6 + p_7) + (p_2 + p_3 + p_5 + p_8)$

$(p_1 + p_4 + p_6 + p_8) + (p_2 + p_3 + p_5 + p_7)$

$(p_1 + p_4 + p_7 + p_8) + (p_2 + p_3 + p_5 + p_6)$

$g = 3$: $[3,3,2]$: $(p_1 + p_2 + p_5) + (p_6 + p_7 + p_8) + (p_3 + p_4)$ $x_1 = 3$, $x_2 = 3$, $x_3 = 2$.

$(p_1 + p_2 + p_6) + (p_5 + p_7 + p_8) + (p_3 + p_4)$

$(p_1 + p_2 + p_7) + (p_5 + p_6 + p_8) + (p_3 + p_4)$

$(p_1 + p_2 + p_8) + (p_5 + p_6 + p_7) + (p_3 + p_4)$

Then

$$B = \frac{1}{2!2!4!}\Big\{(-1)^{8-1}(7!)(12) + (-1)^{8-2}\left[1!5! + 8(2!4!) + 12(3!3!)\right] 12^2 + (-1)^{8-3}(1!2!2!)(12)^3\Big\}$$

A second example involving a general value of n: Find B of the term $Ba_o^{n-6}a_{p_1}a_{p_2}a_{p_3}a_{p_4}a_{p_5}a_{p_6}$ (p_i's $\neq$) where the p_i satisfy the conditions

(9.4) $\quad p_1 + p_2 \equiv p_3 + p_4 \equiv p_5 + p_6 \equiv 0$, $p_1 + p_3 + p_5 \not\equiv p_2 + p_4 + p_6 \equiv 0$

We have:

$g = 1$: $(p_1 + p_2 + p_3 + p_4 + p_5 + p_6)$, $x_1 = 6$

$g = 2$: $\quad (p_1 + p_2) + (p_3 + p_4 + p_5 + p_6), \quad x_1 = 2,\ x_2 = 4$

$\qquad (p_3 + p_4) + (p_1 + p_2 + p_5 + p_6)$

$\qquad (p_5 + p_6) + (p_1 + p_2 + p_3 + p_4)$

$\qquad (p_1 + p_3 + p_5) + (p_2 + p_4 + p_6)$

$g = 3$: $\quad (p_1 + p_2) + (p_3 + p_4) + (p_5 + p_6), \ x_1 = x_2 = x_3 = 2.$

Hence

$$B = (-1)^{6-1} 5!\, n + (-1)^{6-2} \left[3(6) + 2(2)\right] n^2 + (-1)^{6-3} n^3$$

$$B = -120\, n + 22n^2 - n^3$$

There is a certain similarity between Ore's formula for determinant coefficients and (6.9) the formula for permanent coefficients, the partitioning in the determinant case being related to the links of the permanent case. It should be noted that Ore's formula for B is relatively simple to evaluate, but even this would become impractical for larger values of n and most terms.

If a circulant permanent be expressed as $\overset{+}{|} a_{ij} \overset{+}{|}$ then any one of its n! terms $a_{1i} a_{2j} a_{3k} \cdots a_{nm}$ is an even (odd) permutation term if $ijk \cdots m$ is an even (odd) permutation of $123 \cdots n$. And a like statement is of course true for a circulant determinant $|a_{ij}|$.

We can thus write

$$(9.5) \qquad |a_{ij}| = P - N, \qquad \overset{+}{|} a_{ij} \overset{+}{|} = P + N$$

where P, N represent respectively all even and odd permutation terms.

The coefficient B of any particular term T in the <u>determinant</u> can be represented by

$$B = C - D$$

where C, D represent the respective contributions from the even and odd permutations giving T. Now the corresponding term T in the <u>permanent</u> must have the coefficient

$A = C + D$

It has been previously seen that any one of the $n!$ terms in the permanent expansion is $a_{k_0} a_{k_1} \cdots a_{k_{n-1}}$ in the product

$$(a_{k_0} a_{k_1} \cdots a_{k_{n-1}})(x_{i_0} x_{i_1} \cdots x_{i_{n-1}})$$

Also <u>if $i_0 i_1 \cdots i_{n-1}$ is an even (odd) permutation of 0, 1, $\cdots$, n-1 then the associated $a_{k_0} \cdots a_{k_{n-1}}$ is an even (odd) permutation term of the expansion</u>.

But if $i_0 i_1 \cdots i_{n-1}$ in cycle form be represented by $[1^{c_1} 2^{c_2} \cdots n^{c_n}]$, (i.e. c_j cycles of length j), then (Lederman, Introduction to the Theory of Finite Groups)

$$i_0 i_1 \cdots i_{n-1} = \text{even permutation if } n - r \text{ is even,}$$

$$i_0 i_1 \cdots i_{n-1} = \text{odd permutation if } n - r \text{ is odd,}$$

where $r = c_1 + c_2 + \cdots + c_n$.

Furthermore it has been shown in section 2 that the cycle representation of the i-permutation is in 1 - 1 correspondence with the <u>link permutation</u> of the term $a_{k_0} a_{k_1} \cdots a_{k_{n-1}}$ (c_2 links of length 2, c_3 of length 3, $\cdots$, and c_1 = exponent of a_0). Hence $c_2 + c_3 + \cdots + c_n$ equals the <u>number of links</u> in the link-permutation.

It follows that the odd-subscript A's, $A_1, A_3, A_5, \cdots$ contribute entirely to either the even or odd permutation part of A, and $A_2, A_4, A_6, \cdots$ entirely to the opposite type of permutation, i.e.

$$(9.6) \qquad A = (A_1 + A_3 + A_5 + \cdots) + (A_2 + A_4 + A_6 + \cdots) = C + D,$$

and

$$S_1 \equiv A_1 + A_3 + A_5 + \cdots = C,$$
$$S_2 \equiv A_2 + A_4 + A_6 + \cdots = D,$$

or $$S_1 = D,\ S_2 = C.$$

If now $T = Aa_o^{n-m}a_1^{m_1} \cdots$ is a term of the permanent, then $r = (n-m) + (c_2 + c_3 + \cdots + c_n)$, $n - r = m - (c_2 + c_3 + \cdots + c_n)$. Hence, if

m is even, $S_2 = C$, $S_1 = D$, if

m is odd, $S_2 = D$, $S_1 = C$.

If $Ba_o^{n-m}a_1^{m_1} \cdots$ is the corresponding term in the determinant, then we can write

$$B = C - D = (-1)^m (S_2 - S_1),$$

$$A = C + D = S_2 + S_1$$

Or

$$\boxed{\begin{aligned} A &= 2S_1 + (-1)^m B, && (9.7)\\ A &= 2S_2 - (-1)^m B. && (9.8)\end{aligned}}$$

This means that to determine the permanent coefficient A <u>it is sufficient to find the determinant coefficient B and either S_1 or S_2.</u>

Example. Find A in $Aa_o^{n-6}a_{p_1}a_{p_2} \cdots a_{p_6}$, ($p_i \neq$), where the chain-type of the term is

$$\{[p_1p_2][p_3p_4][p_5p_6],\ [p_1p_3p_5][p_2p_4p_6]\}.$$

Direct calculation gives

(9.9) $A_1 = 14n$, $A_2 = 10n^2 - 68n$, $A_3 = n^3 - 12n^2 + 38n$,

$$A = A_1 + A_2 + A_3 = n^3 - 2n^2 - 16n.$$

The evaluation of B was performed above as a previous example with $B = -n^3 + 22n^2 - 120n$. We need now to calculate only $S_2 = A_2 = 10n^2 - 68n$

to find $$A = 2S_2 - (-1)^6 B = 2(10n^2 - 68n) - (-n^3 + 22n^2 - 120n)$$

$$A = n^3 - 2n^2 - 16n$$

as before.

As a check we have $S_1 = A_1 + A_3 = n^3 - 12n^2 + 52n$, and $A = 2S_1 + (-1)^6 B$.

The direct calculation in (9.9) of A_1 and A_3 is much longer and more difficult than that of A_2. By this method here explained we can avoid these difficulties by deriving only A_2, a fairly easy problem.

Further use of this method will be found in the following sections where it is applied to more complicated types of terms from the link-viewpoint.

10. 1-chain and 2-chain terms. General formulas.

The general formula for any 1-chain term has been obtained in section 5 (formula (5.12)). It is derived here again in a simple manner by use of (9.7). If the general 1-chain term be called

$a_o^{n-m} a_{p_1}^{m_1} \cdots a_{p_h}^{m_h}$, so that its chain-type is

$$\left\{ p_1^{m_1} p_2^{m_2} \cdots p_h^{m_h} \right\}$$

then by (9.3)

$$B = \frac{1}{m_1!\,m_2! \cdots m_h!}(-1)^{m-1}(m-1)!\,n \tag{10.1}$$

and by (9.7)

$$A = A_1 = S_1 = -(-1)^m B = \frac{n(m-1)!}{m_1!\,m_2! \cdots m_h!} \tag{10.2}$$

which is the same as given by (5.12).

An example of such a term is $a_o^{19} a_1 a_2 a_3 a_4 a_5^2 a_6$ $(n = 26)$ in which $A = (26)(360)$.

Consider next the 2-chain type terms. The simplest case of this type is

$$\left[p_1^{m_1} p_2^{m_2} \cdots p_g^{m_g} \right] \left[q_1^{n_1} q_2^{n_2} \cdots q_h^{n_h} \right] \equiv [P]\,[Q] \tag{10.3}$$

associated with the term $a_o^{m_o} a_{p_1}^{m_1} \cdots a_{p_g}^{m_g} a_{q_1}^{n_1} \cdots a_{q_h}^{n_h}$

in which no p_i subscript equals a q_j subscript.

Since now $A = A_1 + A_2$ it is sufficient to calculate either A_1 or A_2. A convenient way to obtain A_1 is based on the following familiar Lemma (Uspensky and Heaslet, Elementary Number Theory, McGraw Hill, N.Y., 1939, p. 105).

Suppose we have a collection of objects which may or may not possess one or more of the characteristics $C_1, C_2, \cdots, C_n$, to find how many of the objects do not possess any of the characteristics $C_1, C_2, \cdots, C_n$: Let N be the total number of objects in the collection, $N_1(C_i)$ the number of objects possessing a given character C_i, $N_2(C_i, C_j)$ the number of objects possessing two given characters C_i, C_j, etc. Lemma. The number of objects not possessing any of the characters $C_1, C_2, \cdots, C_n$ is

$$(10.4)\quad N - \sum_i N_1(C_i) + \sum_{ij} N_2(C_i, C_j) - \sum_{ijk} N_3(C_i, C_j, C_k) + \cdots + (-1)^n N_n(C_1, C_2, \cdots, C_n)$$

where the summations are extended over all combinations of the subscripts 1, 2, $\cdots$, n in groups of one, two, three, $\cdots$.

For our present case (10.3) let C_1 represent any permutation of the $L_1 = \sum m_i$ elements $p_1^{m_1}, p_2^{m_2}, \cdots, p_g^{m_g}$ comprising the first link (called the P-link), and C_2 represent any permutation of the second link elements, called the Q-link. Then by the Lemma

$$(10.5)\qquad N_1(\overline{C_1}\overline{C_2}) \equiv N - (N_1(C_1) + N_2(C_2)) + N_2(C_1, C_2)$$

will equal the total number of 1-link permutations. We find

$$N = \frac{L!}{\Pi_P \Pi_Q}, \quad N_1(C_2) = \frac{L_1!}{\Pi_P} \frac{(L_2+1)!}{\Pi_Q},$$

$$(10.6)\qquad N_1(C_2) = \frac{L_2!}{\Pi_Q} \frac{(L_1+1)!}{\Pi_P}, \quad N_2(C_1, C_2) = 2\frac{L_1! L_2!}{\Pi_P \Pi_Q}$$

where

$$L_1 = \sum m_i,\ L_2 = \sum n_i,\ L = L_1 + L_2,$$

(10.7)

$$\Pi P = m_1!m_2! \cdots m_g!,\ \Pi Q = n_1!n_2! \cdots n_h!$$

Hence

(10.8) $$N_1(\overline{C_1C_2}) = \frac{L}{\Pi P\ \Pi Q}\left[(L-1)!-L_1!\,L_2!\right] \equiv T$$

Next, to find A_1, relabel the p and q subscripts as x_1, x_2, $\cdots$, x_{g+h}, and let f_i be the exponent of x_i in the term. Also, let X_i be the number of 1-link permutations beginning with x_i. Then

$$T = X_1 + X_2 + \cdots + X_{g+h},$$

and from (5.10) we see that $A_1 = n\frac{X_i}{f_i}$. Hence $X_i = \frac{A_1}{n} f_i$, so that $T = \frac{A_1}{n}\sum f_i = LA_1/n$. Using this in (10.8) gives finally

(10.9) $$A_1 = \frac{n}{\Pi P\ \Pi Q}\left[(L-1)!\ -L_1!L_2!\right]$$

The coefficient B of (9.3) is easily found to be

$$B = \frac{1}{\Pi P\ \Pi Q}\left\{(-1)^{L-1}(L-1)!n + (-1)^{L-2}(L_1-1)!(L_2-1)!n^2\right\},$$

whence from (10.9) and (9.7), ($S_1 = A_1$), the coefficient A is given by

(10.10) $$\boxed{A = \frac{n}{\Pi P \Pi Q}\left\{(L_1-1)!(L_2-1)!n + (L-1)!-2L_1!L_2!\right\}}$$

We state this as

Theorem 10.1. The coefficient A of the term $a_o^{m_o}a_{p_1}^{m_1} \cdots a_{p_g}^{m_g}a_{q_i}^{n_1} \cdots a_{q_h}^{n_h}$ of chain-type (10.3) in which no $p_i = q_j$ is given by (10.10) (and (10.7)).

Examples of such terms would be $a_o^{13}a_1^2a_3a_4a_7^2a_{17}$, $(n = 20)$, with type $[1^2 4 7^2][3\ 17]$ and $a_o^{12}a_3^2a_9a_{11}a_{12}^2a_{15}^2$, $(n = 20)$, with type $[1^2 4^2 5^2][3\ 17]$.

It is of interest to calculate A_2 directly for the term of

Theorem 10.1. First we have from (9.7) that $A_2 = A_1 + (-1)^L B$ or

$$A_2 = \frac{n(L_1 - 1)!\,(L_2 - 1)!}{\prod P \prod Q}(n - L_1L_2). \tag{10.11}$$

Now let $T(c_1, c_2)$ be the number of distinct 2-link chains each beginning $(p_1, \cdots)(q_1, \cdots)$ and containing c_1, c_2 cycles respectively in each link. Let $n_{c_1}(n'_{c_2})$ be the number of times $p_1(q_1)$ occurs in a cycle. Then

$$(c_1)(n_{c_1}) = m_1 = \text{number of } p_1\text{'s in the chain}$$
$$(c_2)(n'_{c_2}) = n_1 = \text{number of } q_1\text{'s in the chain.}$$

It can further be shown that $N_{a_1}(X_2)$ (see (6.8)) is given by

$$N_{a_1}(X_2) = n - L_1L_2 \tag{10.12}$$

Hence by (6.8) we have

$$A_2 = \sum \frac{n}{c_1c_2} T(c_1, c_2)(n - L_1L_2) = \frac{n(n-L_1L_2)}{m_1n_1} \sum n_{c_1} n'_{c_2} T(c_1, c_2)$$

But this last summation is precisely the total number of 2-link chains beginning $(p_1, \cdots)(q_1, \cdots)$ (they will not all be distinct necessarily), and this number is

$$\frac{(L_1 - 1)!\,(L_2 - 1)!}{\left(\frac{\prod P}{m_1}\right)\left(\frac{\prod Q}{n_1}\right)}$$

Hence

$$A_2 = \frac{n(n-L_1L_2)(L_1-1)!(L_2-1)!}{m_1n_1\left(\frac{\prod P \prod Q}{m_1n_1}\right)} = \frac{n(n-L_1L_2)(L_1-1)!(L_2-1)!}{\prod P \prod Q}$$

agreeing with (10.11).

We come now to the 2-link chain-type terms with a common element (subscript) in the two links. Such a term is denoted by

$$a_o^{m_o} a_r^{u_1+v_1} a_{p_1}^{m_1} \cdots a_{p_g}^{m_g} a_{q_1}^{n_1} \cdots a_{q_h}^{n_h} \tag{10.13}$$

with the chain-type

$$(10.14)\qquad \left[r^{u_1}p_1^{m_1}\cdots p_g^{m_g}\right]\left[r^{v_1}q_1^{n_1}\cdots q_h^{n_h}\right]\equiv\left[r^{u_1}P\right]\left[r^{v_1}Q\right].$$

(First assume P and Q not both vacuous, i.e. assume at least one $m_i > o$ or one $n_i > o$. We indicate this condition by (<u>P or Q not both 0</u>).

It can be shown that now

$$(10.15)\qquad N_{a_1}(X_2) = n - (L_1L_2 - u_1v_1),\left(\begin{matrix}L_1 = u_1 + \sum m_i\\ L_2 = v_1 + \sum n_i\end{matrix}\right),$$

and by an argument similar to that used above we find

$$(10.16)\qquad A_2 = \frac{n}{u_1!\,v_1!\,\prod_P\prod_Q}(L_1-1)!(L_2-1)!\left[n-(L_1L_2-u_1v_1)\right].$$

The value of coefficient B turns out as

$$B = \frac{(-1)^L n}{(u_1+v_1)!\,\prod_P\prod_Q}\left\{\binom{u_1+v_1}{u_1}(L_1-1)!(L_2-1)!\,n-(L-1)!\right\},$$

$$(L = L_1+L_2).$$

from which and (10.16) we easily find

$$(10.17)\qquad \boxed{A = \frac{(L_1-1)!(L_2-1)!}{u_1!\,v_1!\,\prod_P\prod_Q}n^2 + \frac{1}{\prod_P\prod_Q}\left[\frac{(L-1)!}{(u_1+v_1)!}-2\frac{L_1!L_2!}{u_1!\,v_1!}+2\frac{(L_1-1)!(L_2-1)!}{(u_1-1)!(v_1-1)!}\right]n}$$

<u>Theorem 10.2 The coefficient A of the term (10.13) of chain-type (10.14), (P, Q not both 0 and no $p_i = q_j$) is given by (10.17).</u>

In (10.17), $L_1 = u_1 + \sum m_i$, $L_2 = v_1 + \sum n_i$, $L = L_1 + L_2$.

The value of A_1 can also be obtained directly by means of the Lemma and then A found from it, but the method here used is

simpler.

Example of this type is $a_o^{21}a_3^4a_4a_5a_7a_8a_{24}$, $(n = 30)$, with type $[3^2 4\ 5\ 7\ 8][3^2\ 24]$. $A = (30)(1800)$.

In case $P = Q = 0$ the term must be of the form $\underline{a_o^{m_o}a_r^{2u_1}}$, with the chain type $[r^{u_1}][r^{u_1}]$, (so $ru_1 \equiv 0$). In this case $N_{a_1} = n - u_1$, $A_1 = 0$, $A_2 = (-1)^L B$, and there results

$$(10.18) \qquad \boxed{A = \frac{n}{2!\, u_1^2}\ (n-u_1)}$$

This type term is also considered in a later section. The term $a_o^{15}a_5^{10}$, $(n = 25)$, illustrates this type, $A = 10$.

11. <u>2-chain terms, continued</u>. In this section we give a formula for the coefficient A of the most general chain-type $[U]\,[V]$ where U and V can involve any number of common elements (case of one commom element given by (10.17)). The situation with two common elements is described first, and then the general case is given.

Since a long and involved analysis is required only an outline of the various steps involved will be set down.

Consider then a term with the chain type

$$(11.1) \quad [U][V] \equiv [r^{u_1}s^{u_2}p_1^{m_1}\cdots\ p_g^{m_g}]\ \ [r^{v_1}s^{v_2}q_1^{n_1}\ \cdots\ q_h^{n_h}]$$
$$= [r^{u_1}s^{u_2}P]\,[r^{v_1}s^{v_2}Q]$$

where again no $p_i = q_j$. Assume first the links are not identical, (i.e. exclude the case $P = Q = 0$, $u_1 = v_1$, $u_2 = v_2$).

The determination of N_{a_1} for any particular link-permutations is now more difficult since N_{a_1} will in general vary from one such permutation to another.(Note that in (10.12) and (10.15), N_{a_1} is independent of the link-permutation $X_2 = P_1P_2$).

Let $y_1 y_2 \cdots y_u$ be any permutation of elements r and s selected from U and $z_1 z_2 \cdots z_u$ be such a permutation selected from V. Then if

(11.2) $\quad y_1 \neq z_1,\ y_1 y_2 \neq z_1 z_2, \cdots,\ y_1 y_2 \cdots y_{u-1} \neq z_1 z_2 \cdots z_{u-1}$

but

(11.3) $\quad y_1 y_2 \cdots y_u = z_1 z_2 \cdots z_u$

the relation (11.3) is called an <u>identity-sequence, of length u</u>. For example (ssrsrrrsssrr, rrsrsssrrrss) is <u>not</u> an identity sequence since ssrsrr = rrsrss although s $\neq$ r, ss $\neq$ rr, ssr $\neq$ rrs, ssrs $\neq$ rrsr, ssrsr $\neq$ rrsrs.

However, (ssrsrssrrsrr, rrsrsrrssrss) is an identiy-sequence of length 12. The inequality $y_1 y_2 \cdots y_w \neq z_1 z_2 \cdots z_w$ means the number of r's in the two permutations $y_1 \cdots y_w$ and $z_1 \cdots z_w$ is different (and of course for s's also). The equality (11.3) means both permutations have the same number of r's and s's.

Now it can be shown that the value of N_{a_1} for any particular link-permutation obtainable from (11.1) is given by

(11.4) $\quad N_{a_1} = n-(L_1 L_2 - \Sigma\, t_i)$

where t_i is the total number of identity-sequences of length i, the first half of the sequence contained in the first link, and the second half in the second link, and the link elements are considered cyclically in this count.

Thus in the link-permutation

$(\underline{s}, r, r, r, s, r, p_1, \underline{r})(\underline{r}, r, r, s, q_1, \underline{s})$

we find $t_1 = (2)(1) + (5)(3) = 17$,

$t_2 = 3,\ t_3 = 1,\ t_4 = 1$

For t_2 we count the (rs,sr) composed of the underlined elements

considered cyclically.

Hence $N_{a_1} = n - [(8)(6) - (17 + 3 + 1 + 1)] = n - 26.$

It is next shown that A_2 is expressible as

$$A_2 = \frac{n}{u_1 v_2} T \tag{11.5}$$

where $T = T_o + T_1 + T_2 + \cdots$, and the T_i are obtained as follows:

Let $I(i_1, i_2)$ be the number of different types of identity-sequences of a given length $x = i_1 + i_2$, where i_1, i_2 are the respective number of r's and s's in each half of a sequence.

Thus,

$$I(1, 0) = I(0, 1) = 1 \quad ((r, r), (s, s))$$

$$I(1, 1) = 2, \quad ((rs, sr), (sr, rs))$$

$$I(2, 1) = 2, \quad ((rrs, srr), (srr, rrs)), \text{ etc.}$$

It is shown that

$$I(i_1, i_2) = I(i_2, i_1) = 2\,\frac{i_1 + i_2 - 1}{i_1 i_2} \binom{i_1 + i_2 - 2}{i_2 - 1}\binom{i_1 + i_2 - 2}{i_1 - 1}, \tag{11.6}$$

$$(i_1 + i_2 \geq 2,\ i_1 i_2 \neq 0).$$

Then T_x is given by

$$T_x = u_1 v_2 \sum_{i_1 + i_2 = x} I(i_1, i_2)\, M_{i_1 i_2}, \quad (x \geq 1) \tag{11.7}$$

where

$$M_{i_1 i_2} = \frac{(L_1 - (i_1 + i_2))!\,(L_2 - (i_1 + i_2))!}{\prod_P \prod_Q (u_1 - i_1)!\,(u_2 - i_2)!\,(v_1 - i_1)!\,(v_2 - i_2)!}, \tag{11.8}$$

$$(\text{all } i_1, i_2).$$

$$T_o = \frac{(n - L_1 L_2)(L_1 - 1)!\,(L_2 - 1)!}{\prod_P \prod_Q (u_1 - 1)!\,u_2!\,v_1!\,(v_2 - 1)!} \tag{11.9}$$

Finally by (11.5), (11.7), and $T = \sum T_x$, we have

$$(11.10) \qquad A_2 = \frac{n^2(L_1 - 1)!(L_2 - 1)!}{\Pi} + n \sum_{i_1+i_2=0} I(i_1, i_2)\, M_{i_1 i_2},$$

where $\Pi = u_1!\, u_2!\, v_1!\, v_2!\, \Pi_P\, \Pi_Q$, $I(0, 0) = -1$, $I(0, 1) = I(1, 0) = 1$. The summation in A_2 proceeds: $(I(0, 0)\, M_{00})$ +

$(I(0, 1)M_{01} + I(1, 0)M_{10}) + (I(1, 1)M_{11}) + (I(2, 1)M_{21} + I(1, 2)M_{12})$ + ··· till a zero term is reached.

T_x represents the number of all x-length identity-sequences obtainable from <u>all</u> link-permutations starting $(r, \cdots)(s, \cdots)$, $(x > 0)$.

The expression $u_1 v_2 M_{i_1 i_2}$ represents the number of times each identity-sequence of given length $x = i_1 + i_2$ (each half containing i_1 r's, i_2 s's) appears in <u>all</u> link-permutations starting $(r, \cdots)(s, \cdots)$.

Finally the coefficient B is found as

$$(11.11) \qquad B = \frac{1}{(u_1+v_1)!(u_2+v_2)!\ \Pi_P\, \Pi_Q} \Big[(-1)^{L-1}(L-1)!n + (-1)^{L-2} \binom{u_1+v_1}{u_1} \binom{u_2+v_2}{u_2} (L_1-1)!(L_2-1)!n^2 \Big]$$

The value of coefficient is given in the Theorem below.

<u>Theorem 11.1. The coefficient A of the term</u>

$$a_o^{m_o} a_r^{u_1 + v_1} a_s^{u_2 + v_2} a_{p_1}^{m_1} \cdots a_{p_g}^{m_g} a_{q_1}^{n_1} \cdots a_{q_h}^{n_h}$$

<u>of chain-type (11.1), (with $p_i \neq q_j$ and excluding the case $P = Q = 0$, $u_1 = v_1$, $u_2 = v_2$) is given by</u>

$$A = \frac{(L_1 - 1)!(L_2 - 1)!}{u_1!u_2!v_1!v_2!\Pi_P\ \Pi_Q} n^2$$

$$(11.12) \qquad + \left[\frac{(L - 1)!}{(u_1+v_1)!(u_2+v_2)!\,\Pi_P\, \Pi_Q} + 2 \sum_{i_1+i_2=0} I(i_1, i_2) M_{i_1 i_2} \right] n$$

where $I(i_1, i_2)$ is defined by (11.6) with $I(0, 0) = -1$, $I(1,0) = I(0, 1) = 1$ and $M_{i_1 i_2}$ by (11.8), $\left(L_1 = u_1 + u_2 + \sum m_i,\ L_2 = v_1 + v_2 + \sum n_i,\ L = L_1 + L_2\right)$.

The exceptional case $P = Q = 0$, $u_1 = v_1$, $u_2 = v_2$ gives the chain-type $[r^{u_1} s^{u_2}][r^{u_1} s^{u_2}]$. It can be shown now that $A_1 = 0$ (no 1-link permutation possible). Hence we find easily

$$\text{(11.13)} \quad \boxed{A = A_2 = \frac{1}{2(u_1+u_2)^2}\binom{u_1+u_2}{u_1}^2 n^2 - \frac{1}{2(u_1+u_2)}\binom{2u_1+2u_2}{2u_1} n}$$

Illustrations of these types are

$a_0^{13} a_2^3 a_3^3 a_6 a_8 a_{15}$ $(n = 22)$, with chain-type $[3^2 2\ 6\ 8][3\ 2^2 15]$, and

$a_0^{14} a_{10}^6 a_{15}^4$ $(n = 24)$, with chain-type $[3^2 2^3][3^2 2^3]$.

We come now to the general 2-chain type term having any number of common elements in each link. Such a term is represented by the chain-type

$$\text{(11.14)} \quad \left[r_1^{u_1} r_2^{u_2} \cdots r_m^{u_m} P\right]\left[r_1^{v_1} r_2^{v_2} \cdots r_m^{v_m} Q\right]$$

where P, Q have the same meanings as in (11.1).

An expression is first found for A_2 analogous to (11.10) where $m = 2$. The functions corresponding to $I(i_1, i_2)$ will be given in terms of a difference equation (recurrent formula).

First define a function $J_x(i_1, i_2, \cdots, i_m; j_1, \cdots, j_m)$. Let $(y_1 y_2 \cdots y_x, z_1 z_2 \cdots z_x)$ be a pair of permutations consisting respectively of i_1 r_1's, i_2 r_2's, $\cdots$, i_m r_m's and j_1 r_1's, $\cdots$, j_m r_m's, with $\sum i = \sum j = x$. The function $J_x(i_1, \cdots, i_m; j_1, \cdots, j_m)$ equals the number of pairs of such permutations with the property

(11.15) $y_1 \neq z_1$, $y_1 y_2 \neq z_1 z_2$, $y_1 y_2 y_3 \neq z_1 z_2 z_3$, $\cdots$, [illegible]

$$y_1 y_2 \cdots y_x \neq z_1 z_2 \cdots z_x.$$

Example: $(r_1r_1r_3r_2,\ r_2r_1r_3r_2)$, $(x = 4,\ m = 3,\ i_1 = 2,\ i_2 = i_3 = 1;$ $j_1 = j_3 = 1,\ j_2 = 2)$.

Then a simple analysis shows that

$$(11.16)\quad J_{x+1}(i_1, i_2, \cdots, i_m; j_1, \cdots, j_m) = \sum_{\alpha,\beta=1}^{m} J_x(i_1, \cdots, i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1}, \cdots i_m; j_1, \cdots, j_{\beta-1}, j_\beta - 1, j_{\beta+1}, \cdots, j_m)$$

with the conditions $\sum i = \sum j = x + 1$. Also

$J_{x+1}(i_1, \cdots, i_m; i_1, \cdots, i_m) = 0.$

Now define a function $I_{x+1}(i_1, i_2, \cdots, i_m; r_\alpha, r_\beta)$, $(\alpha \neq \beta)$, by the conditions

$I_{x+1}(i_1, i_2, \cdots, i_m; r_\alpha, r_\beta)$ = the number of pairs of permutations $(y_1y_2 \cdots y_xr_\alpha,\ z_1z_2 \cdots z_xr_\beta)$ such that $y_1 \neq z_1,\ y_1y_2 \neq z_1z_2, \cdots, y_1 \cdots y_x \neq z_1 \cdots z_x,\ y_1 \cdots y_xr_\alpha = z_1 \cdots z_xr_\beta$ where $y_1 \cdots y_xr_\alpha$ and $z_1 \cdots z_xr_\beta$ both consist of $i_1\,r_1$'s, $i_2\,r_2$'s, $\cdots, i_m\,r_m$'s, $(\sum i = x + 1)$.

Example: $(r_1r_3r_3r_2\underline{r_1},\ r_2r_1r_3r_1\underline{r_3})$ as one pair for $I_5(2, 1, 2; r_1, r_3)$.

Note that I_{x+1} is a generalization of $I(i_1, i_2)$ of the case $m = 2$.

It will then follow that

$$(11.17)\quad I_{x+1}(i_1, \cdots, i_m; r_\alpha, r_\beta) = J_x(i_1, \cdots, i_{\alpha-1}, i_\alpha - 1, i_{\alpha+1}, \cdots, i_m; i_1, \cdots, i_{\beta-1}, i_\beta - 1, i_{\beta+1}, \cdots, i_m),\ (\textstyle\sum i = x + 1).$$

Next put

$$(11.18)\quad M(i_1, i_2, \cdots, i_m) = \frac{(L_1 - \sum i)!\,(L_2 - \sum i)!}{\prod_\rho \prod_Q (u_1 - i_1)! \cdots (u_m - i_m)!\,(v_1 - i_1)! \cdots (v_m - i_m)!}$$

Then

$$A_2 = \frac{(L_1-1)!(L_2-1)!}{\Pi} n^2 + \left\{\sum I(i_1, i_2, \cdots, i_m; r_\alpha, r_\beta) M(i_1, i_2, \cdots, i_m)\right\} n \tag{11.19}$$

where $I(0, \cdots, 0) = -1$, $I(0, \cdots, 0, 1, 0, \cdots, 0) = 1$, (no r_α, r_β present), and the summation covers all possible values of $i_1, i_2, \cdots, i_m$ and α, β with $\alpha \neq \beta$. Note the index $x + 1$ on the I function is here unnecessary due to the summation.

In addition

$$\Pi = u_1! \cdots u_m!\, v_1! \cdots v_m!\, \Pi_P\, \Pi_Q,\quad L_1 = \sum u_i + \sum m_i,\quad L_2 = \sum v_i + \sum n_i,$$

$L = L_1 + L_2$.

The coefficient B is found without difficulty and finally the coefficient A is obtained as

$$\boxed{A = \frac{(L_1-1)!(L_2-1)!}{\Pi} n^2 + \left\{\frac{(L-1)!}{(u_1+v_1)! \cdots (u_m+v_m)!\,\Pi_P\,\Pi_Q} + 2\sum I(i_1, i_2, \cdots, i_m; r_\alpha, r_\beta) M(i_1, \cdots, i_m)\right\} n} \tag{11.20}$$

The exceptional case $u_i = v_i$, $P = Q = 0$ with chain-type $[r_1^{u_1} \cdots r_m^{u_m}]\,[r_1^{u_1} \cdots r_m^{u_m}]$ gives the value for A as

$$A = \frac{1}{2}\left[\frac{(L-1)!}{u_1! \cdots u_m!}\right]^2 n^2 + \left[\frac{(2L-1)!}{(2u_1)! \cdots (2u_m)!} + \sum I(i_1, i_2, \cdots, i_m; r_\alpha, r_\beta) M(i_1, \cdots, i_m) - \sum \frac{i_k}{u_k} I(i_1, \cdots, i_m; r_\alpha, r_\beta) M(i_1, \cdots, i_m)\right] n, \tag{11.21}$$

$$(L = \sum u).$$

where the subscript k in the second summation term can have any of the values 1, 2, $\cdots$, m but must of course be kept fixed throughout the summation.

To convert (11.21) to a symmetric form the factor i_k/u_k in the

second summation can be replaced by $(i_1 + \cdots + i_m)/L$, though the given form is easier to use.

Several identities connecting the I functions together with a large number of its numerical values are given in the Summary where also are included some worked out formulas for A.

12. 2-chain terms (concluded). In this section some formulas are given for coefficients of terms whose chain-types involve two or more pairs of 2-chains. The simplest such case is represented by the chain-type

$$\left\{ \left[G_1G_2\right] \left[G_3G_4\right],\ \left[G_1G_3\right] \left[G_2G_4\right] \right\} \tag{12.1}$$

where

$$G_1 = p_1^{m_1} \cdots p_g^{m_g},\ G_2 = q_1^{n_1} \cdots q_h^{n_h},\ G_3 = r_1^{u_1} \cdots r_i^{u_i},\ G_4 = s_1^{v_1} \cdots s_j^{v_j} \tag{12.2}$$

and there are no common elements between any two G's.

The value of coefficient A is

$$\boxed{\begin{aligned} A = {} & \frac{1}{\prod G} \left[(L_1-1)!(L_2-1)! + (M_1-1)!(M_2-1)!\right] n^2 \\ & + \frac{1}{\prod G} \left[(L-1)! - 2L_1!\ L_2! - 2M_1!M_2! \right. \\ & \left. \qquad + 4g_1!\ g_2!\ g_3!\ g_4!\right] n \end{aligned}} \tag{12.3}$$

in which

$$g_1 = m_1 + \cdots + m_g,\ g_2 = n_1 + \cdots + n_h,$$
$$g_3 = u_1 + \cdots + u_i,\ g_4 = v_1 + \cdots + v_j,$$

$$L_1 = g_1 + g_2,\ L_2 = g_3 + g_4,\ M_1 = g_1 + g_3,\ M_2 = g_2 + g_4, \tag{12.4}$$
$$\prod G = (m_1!\cdots m_g!)(n_1!\cdots n_h!)(u_1!\cdots u_i!)(v_1!\cdots v_j!)$$

An illustration is $a_0^6 a_1 a_2 a_3 a_4 a_5 a_9$, $(n = 12)$

Here the type is $\{[1\ 2\ 4\ 5][9\ 3],\ [1\ 2\ 9]\ [4\ 5\ 3]\}$, $G_1 = 1\ 2$,

$G_2 = 4\ 5$, $G_3 = 9$, $G_4 = 3$,

$g_1 = g_2 = 2$, $g_3 = g_4 = 1$, $L_1 = 4$, $L_2 = 2$, $M_1 = 3$, $M_2 = 3$, $L = 6$, and

$A = (12)(88)$.

The next type is one with three pairs of 2-chains

(12.5) $\{[G_1G_2G_3]\ [G_4G_5],\ [G_1G_3G_4]\ [G_2G_5],\ [G_2G_3G_4]\ [G_1G_5]\}$

The first four G's given by (12.2) and G_5 by

$$G_5 = t_1^{w_1} t_2^{w_2} \cdots t_k^{w_k}$$

Again we assume <u>no common elements between any two of the G's</u>.

The formula for A turns out to be

(12.6)
$$\boxed{A = \frac{1}{\prod G} \Big\{ \big[(L_1-1)!(L_2-1)! + (M_1-1)!(M_2-1)! + (N_1-1)!(N_2-1)!\big]\ n^2 + \big[(L-1)! - 2(L_1!\,L_2! + M_1!M_2! + N_1!N_2!) + 2E\big]\ n \Big\}}$$

in which

$$E = g_1!g_2!g_3!g_4!g_5! \left[\binom{g_1+g_3}{g_1} + \binom{g_2+g_3}{g_2} + \binom{g_3+g_4}{g_3}\right],$$

(12.7) $g_5 = w_1 + \cdots + w_k$, $L_1 = g_1 + g_2 + g_3$, $L_2 = g_4 + g_5$, $M_1 = g_1 + g_3 + g_4$,

$M_2 = g_4 + g_5$, $N_1 = g_2 + g_3 + g_4$, $N_2 = g_1 + g_5$, $L = L_1 + L_2$,

and $\prod_G$ is defined as in (12.4) with the additional factors $(w_1! \cdots w_k!)$.

Example: $a_0^{13}a_1a_2a_8a_9a_{11}a_{14}a_{15}$, $(n = 20)$, type

$\{[1\ 8\ 9\ 2]\ [14\ 15\ 11],\ [1\ 8\ 2\ 14\ 15]\ [9\ 11],\ [9\ 2\ 14\ 15]\ [1\ 8\ 11]\}$

$G_1 = 1\ 8$, $G_2 = 9$, $G_3 = 2$, $G_4 = 14\ 15$, $G_5 = 11$.

When common elements are present in the various links the calculations are more difficult. We give the simplest such type only:

$$(12.8)\left\{\left[r^{w_1+w_2}G_1G_2\right]\left[r^{w_3+w_4}G_3G_4\right],\left[r^{w_1+w_3}G_1G_3\right]\left[r^{w_2+w_4}G_2G_4\right]\right\}$$

with the G's as in (12.2) and possessing no common element between themselves.

The value of A is

$$\boxed{\begin{aligned} A = {} & \frac{1}{\prod G}\left[\frac{(L_1-1)!(L_2-1)!}{(w_1+w_2)!(w_3+w_4)!}+\frac{(M_1-1)!(M_2-1)!}{(w_1+w_3)!(w_2+w_4)!}\right]n^2 \\ (12.9)\quad & +\frac{1}{\prod G}\left[\frac{(L-1)!}{W!}-2\frac{L_1!L_2!}{(w_1+w_2)!(w_3+w_4)!}-2\frac{M_1!M_2!}{(w_1+w_3)!(w_2+w_4)!}\right. \\ & +2\frac{(L_1-1)!(L_2-1)!}{(w_1+w_2-1)!(w_3+w_4-1)!}+2\frac{(M_1-1)!(M_2-1)!}{(w_1+w_3-1)!(w_2+w_4-1)!} \\ & \left. +4\sum_{x=0} E(x)\right]n \end{aligned}}$$

and

$$\begin{aligned} E(x) = {} & g_1!g_2!g_3!g_4!\left[\binom{x+g_1}{g_1}\binom{x+w_4-w_1+g_4}{g_4}-2\binom{x-1+g_1}{g_1}\binom{x-1+w_4-w_1+g_4}{g_4}\right. \\ (12.10)\quad & \left.+\binom{x-2+g_1}{g_1}\binom{x-2+w_4-w_1+g_4}{g_4}\right]\left[\binom{w_1+w_2-x+g_2}{g_2}\binom{w_1+w_3-x+g_3}{g_3}\right], \end{aligned}$$

$$\begin{aligned} & L_1 = w_1+w_2+g_1+g_2,\ L_2 = w_3+w_4+g_3+g_4,\ M_1 = w_1+w_3+g_1+g_3, \\ (12.11)\quad & M_2 = w_2+w_4+g_2+g_4,\ L = L_1+L_2,\ W = w_1+w_2+w_3+w_4, \end{aligned}$$

Also the notation is chosen so that $w_1 \leq w_4$.

Example: $a_0^{10}a_1^7a_2a_{15}a_6$ $(n = 20)$, with type $\left\{\left[1^3 2\ 15\right]\left[1^4 16\right],\left[1^2 2\ 16\right]\left[1^5 15\right]\right\}$.

Here $w_1 = 2$, $w_2 = 1$, $w_3 = 0$, $w_4 = 4$; $G_1 = 2$, $G_2 = 15$, $G_3 = 16$, $G_4 = 0$.

13. <u>3-link and 4-link chain terms</u>.

The simplest 3-link chain type <u>without common elements between links</u> is represented by

$$(13.1)\qquad [G_1]\,[G_2]\,[G_3]$$

still using the notation of (12.2) for the elements of the links. It is instructive to calculate A in two ways: first by finding A_1 and A_3 giving $S_1 = A_1 + A_3$, and second by finding $A_2 = S_2$.

To find A_1 we use the Lemma.

Let G_i = any permutation of elements of link (G_i),

G_{ij} = any permutation of the elements of the combined links (G_i), (G_j),

$L_i = g_i$, $g_1 = m_1 + \cdots + m_g$, $g_2 = n_1 + \cdots + n_h$, $g_3 = u_1 + \cdots + u_i$.

Then

$$(13.2) \qquad N_1(G_i) = \frac{L_i!(L-L_i+1)!}{\prod G_i}, \quad \left(\begin{array}{c}\prod G_1 = m_1!m_2! \cdots m_g!, \text{ etc.,} \\ L = L_1 + L_2 + L_3\end{array}\right)$$

$$(13.3) \qquad N_1(G_{ij}) = \frac{(L_k+1)!n_{ij}}{\prod G_k}, \quad (i, j, k = 1, 2, 3 \text{ in any order}),$$

where

$$(13.4) \qquad n_{ij} = \frac{1}{\prod G_i \prod G_j}\left[(L_i + L_j)! - (L_i + L_j)L_i!L_j!\right]$$

gives the <u>total</u> number of 1-link-permutations made up from the elements of links (G_i) and (G_j) combined, i.e. n_{ij} is the subset of all the permutations G_{ij} which are 1-chains (first element not kept fixed in this count to give n_{ij}).

Further,

$$(13.5) \qquad N_2(G_i, G_j) = \frac{L_i!L_j!(L_k+2)!}{\prod G}, \quad \left(\prod G = \prod G_1 \prod G_2 \prod G_3\right),$$

$$(13.6) \qquad N_2(G_i, G_{jk}) = \frac{2!L_i!n_{jk}}{\prod G_i},$$

$$(13.7) \qquad N_3(G_1, G_2, G_3) = \frac{3!L_1!L_2!L_3!}{\prod G}$$

Hence the <u>total</u> number of 1-link chains (not keeping first element fixed) is

$$(13.8)\quad n_{123} = N - \sum\left[N_1(G_i) + N_1(G_{ij})\right] + \sum\left[N_2(G_i, G_j) + N_2(G_i, G_{jk})\right] - N_3(G_1, G_2, G_3),$$

or

$$(13.9)\quad n_{123} = \frac{L}{\prod G}\left[(L-1)! + (L+1)L_1!L_2!L_3! - L!\sum \frac{1}{\binom{L}{L_i}}\right]$$

It follows then that

$$(13.10)\quad A_1 = \frac{n}{L}(n_{123})$$

Next A_3 is found. Let

$$(13.11)\quad (p_1', p_2', \cdots, p_{L_1}')(q_1', \cdots, q_{L_2}')(r_1', \cdots, r_{L_3}')$$

be any particular 3-link chain, and put

$$s_i' = p_0' + p_1' + \cdots + p_i',\ t_i' = q_0' + q_1' + \cdots + q_i',$$

(13.12)

$$u_i' = r_0' + r_1' + \cdots + r_i',\ (p_0' = q_0' = r_0' = 0).$$

(see, e.g., (3.3)).

The parameter-values a, b must satisfy

$$(13.13)\quad a \not\equiv s_i' - t_k',\ b \not\equiv s_j' - u_i',\ b \not\equiv a + (t_k' - u_m'),\ (\text{all } i,j,k,m).$$

There are $(n-L_1L_2)$ allowable a-values and a <u>minimum</u> of $n-(L_1L_3+L_2L_3)$ allowable b-values for <u>each</u> a-value. For a given a-value the excluded b's may not all be distinct. There will be duplicate b's whenever

$$(13.14)\quad a + (t_k' - u_m') \equiv (s_j' - u_i')$$

In (13.14) if $i = m$ there results $a \equiv s_j' - t_k'$ contradicting (13.13). If $i \neq m$ we write

$$(13.15)\quad a \equiv (s_j' - u_i') + (u_m' - t_k').$$

This latter equation <u>always</u> has a solution for parameter-value a for all values of i, j, k, m, $(i \neq m)$. Because if not we would have

(13.16) $(s_j' - u_i') + (u_m' - t_k') \equiv s_e' - t_f'$ (for some e, f)

or

(13.17) $(s_j' - s_e') + (t_j' - t_k') + (u_m' - u_i') \equiv 0,$

which contradicts the assumption of chain-type (13.1)

The number of duplicate b-values will therefore equal the number of a-solutions to (13.14), $(i \neq m)$. The number of choices for $(s_j' - u_i')$ is L_1L_3 and then for $(t_k' - u_m')$ the number is $L_2(L_3-1)$, or a total of $L_1L_2L_3(L_3-1)$.

The value of N_{ab} is then

(13.18) $$N_{ab} = (n-L_1L_2)(n-(L_1L_3 + L_2L_3))+L_1L_2L_3(L_3-1),$$

and this value remains the same for any 3-chain (13.11). The value of A_3 must thus be

(13.19) $$A_3 = \frac{n}{\Pi G}(L_1-1)!(L_2-1)!(L_3-1)!\,N_{ab}$$

(See the discussion following (10.11)).

The value of A is then determined after finding coefficient B. (A is given in the Summary).

Next, to find A_2. To form a 2-link chain two of the three links (G_1), (G_2), (G_3) must be combined to form a single link. The number of 1-link chains thus obtainable from links (G_i), (G_j) is n_{ij} of (13.4). Denote the corresponding 2-link chain by $(\overline{G_iG_j})(G_k)$. The value of N_a for this 2-chain is $n-L_k(L_i + L_j)$, so that

(13.20) $$A_2 = n\sum\left(\frac{n_{ij}}{L_i + L_j}\right)\frac{(L_k-1)!}{\Pi G_k}\left[n-L_k(L_i + L_j)\right],$$

(13.21) $$A_2 = \frac{1}{\Pi G}\left\{n^2\sum(L_k-1)!\left[(L_i + L_j-1)!-L_i!L_j!\right]\right.$$
$$\left.-n\left[\sum L_k!\,(L_i + L_j)!\,-2L(L_1!L_2!L_3!)\right]\right\}$$

In the summations $k = 1, 2, 3$ with respective values of $ij = 2\ 3, 3\ 1, 1\ 2$.

The coefficients for the chain-type

$$(13.22) \quad \left[r^{u_1}G_1\right]\left[r^{u_2}G_2\right]\left[r^{u_3}G_3\right],$$

and

$$(13.23) \quad \left\{\left[G_1G_2\right]\left[G_3G_4\right]\left[G_5G_6\right], \left[G_1G_3G_5\right]\left[G_2G_4G_6\right]\right\}$$

have also been worked out, and the results will be found in the Summary.

The simplest case of a 4-chain type

$$(13.24) \quad \left[G_1\right]\left[G_2\right]\left[G_3\right]\left[G_4\right]$$

will also be found in the Summary.

14. Application of Muir's Theorem. One of Muir's theorems is useful for finding the coefficients of certain types of terms (T. Muir, Question 6001 Educ. Times lxv (1912), p. 139). We state this theorem here without proof.

Theorem 14.1. In the expansion of the product $u_1u_2 \cdots u_n$ where $u_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$, the coefficient of $x_1^{e_1}x_2^{e_2} \cdots x_n^{e_n}$ is given by the permanent

$$(14.1) \quad \frac{1}{e_1!e_2!\cdots e_n!}\begin{vmatrix} a_{11}\cdots a_{11} & a_{12}\cdots a_{12} & \cdots & a_{1n}\cdots a_{1n} \\ a_{21}\cdots a_{21} & a_{22}\cdots a_{22} & \cdots & a_{2n}\cdots a_{2n} \\ a_{31}\cdots a_{31} & a_{32}\cdots a_{32} & \cdots & a_{3n}\cdots a_{3n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}\cdots a_{n1} & a_{n2}\cdots a_{n2} & \cdots & a_{nn}\cdots a_{nn} \\ \longleftarrow e_1 \longrightarrow & \longleftarrow e_2 \longrightarrow & \cdots & \longleftarrow e_n \longrightarrow \end{vmatrix}^{+}$$

where there are e_i identical columns $a_{1i}a_{2i} \cdots a_{ni}$, $(\Sigma e_i = n)$.

Note that if all $e_i = 1$ we have the definition of a permanent $\overset{+}{|}a_{ij}\overset{+}{|}$.

Suppose now the a_{ij} are the elements of a general circulant permanent $C_n(0, 1, \cdots, n-1)$. Then Theorem 14.1 states that <u>the coefficient A of the term</u>

$$Aa_o^{e_o} a_1^{e_1} \cdots a_{n-1}^{e_{n-1}}$$

<u>in the expansion of C_n is given by</u>

<u>A = coefficient of $(a_o a_1 \cdots a_{n-1})$ in</u>

(14.2)

$$\frac{1}{e_o! e_1! \cdots e_{n-1}!}\ \overset{+}{\left|\begin{array}{ccccc} a_o \cdots a_o & a_1 \cdots a_1 & a_2 \cdots a_2 & \cdots & a_{n-1} \cdots a_{n-1} \\ a_1 \cdots a_1 & a_2 \cdots a_2 & a_3 \cdots a_3 & \cdots & a_o \cdots a_o \\ a_2 \cdots a_2 & a_3 \cdots a_3 & a_4 \cdots a_4 & \cdots & a_1 \cdots a_1 \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot & \cdot \quad \cdot \\ \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot & \cdot & \cdot \quad \cdot \\ a_{n-1} \cdots a_{n-1} & a_o \cdots a_o & a_1 \cdots a_1 & \cdots & a_{n-2} \cdots a_{n-2} \\ \longleftarrow e_o \longrightarrow & \longleftarrow e_1 \longrightarrow & \longleftarrow e_2 \longrightarrow & \cdots & \longleftarrow e_{n-1} \longrightarrow \end{array}\right|}^{+}$$

(In the Theorem the roles of a and x have been interchanged)

(A) <u>Two-factor terms</u>. This latter result is now applied to obtain the coefficient of any term.

(14.3) $$Aa_o^{n-r} a_p^r .$$

Such a term is called a <u>two-factor term</u> (factors a_o, a_p).

From the above result we will have then that

A = coefficient of $(a_o a_1 \cdots a_{n-1})$ in

$$\frac{1}{(n-r)!\,r!}\;\begin{vmatrix} a_0 & \cdots & a_0 & a_p & \cdots & a_p \\ a_1 & \cdots & a_1 & a_{p+1} & \cdots & a_{p+1} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ a_{n-1} & \cdots & a_{n-1} & a_{p-1} & \cdots & a_{p-1} \end{vmatrix}^{+}_{\longleftarrow n-r \longrightarrow \longleftarrow r \longrightarrow} \qquad \text{or} \tag{14.4}$$

A = coefficient of $(a_0 a_1 \cdots a_{n-1})$ in

$$\frac{a_0 a_1 \cdots a_{n-1}}{(n-r)!\,r!}\;\begin{vmatrix} 1 & \cdots & 1 & \frac{a_p}{a_0} & \cdots & \frac{a_p}{a_0} \\ 1 & \cdots & 1 & \frac{a_{p+1}}{a_1} & \cdots & \frac{a_{p+1}}{a_1} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ 1 & \cdots & 1 & \frac{a_{p-1}}{a_{n-1}} & \cdots & \frac{a_{p-1}}{a_{n-1}} \end{vmatrix}^{+}_{\longleftarrow n-r \longrightarrow \longleftarrow r \longrightarrow} \equiv \frac{a_0 a_1 \cdots a_{n-1}}{(n-r)!\,r!}\,D, \tag{14.5}$$

where D is the permanent thus defined.

Hence

$$A = \frac{1}{(n-r)!\,r!}(\text{constant term of } D). \tag{14.6}$$

Expand D by Laplace's expansion using the first (n-r) columns against the last r columns. All first (n-r)-column cofactors are

$$\begin{vmatrix} 1 & \cdots & 1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & \cdots & 1 \end{vmatrix}^{+}_{n-r} = (n-r)!$$

The last r-column cofactors are of type

$$(14.7)\qquad \left|\begin{matrix} \frac{a_i}{a_{i-p}} & \cdots & \frac{a_i}{a_{i-p}} \\ \frac{a_j}{a_{j-p}} & \cdots & \frac{a_j}{a_{j-p}} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{a_k}{a_{k-p}} & \cdots & \frac{a_k}{a_{k-p}} \end{matrix}\right|_r^{+} = r!\left(\frac{a_i}{a_{i-p}}\ \frac{a_j}{a_{j-p}} \cdots \frac{a_k}{a_{k-p}}\right) \equiv r!T_{ij\cdots k}$$

where $ij \cdots k$ is any combination of r numbers selected from 0, 1, $\cdots$, $n-1$, and $T_{ij\cdots k}$ is defined as shown.

It is evident then that

A = the number of $T_{ij\cdots k}$ which are constant (and therefore equal to 1).

From the form of D it follows that an equivalent formulation of the problem of finding A is this:

A = the number of ways of selecting r of the n fractions

$$(14.8)\qquad \frac{a_p}{a_0}, \frac{a_{p+1}}{a_1}, \frac{a_{p+2}}{a_2}, \ldots, \frac{a_{p-1}}{a_{n-1}}$$

so their product equals 1.

Now the only way such a product of r factors can equal 1 is for this product to factor into sets of the form

$$P_m = \frac{a_{i+p}}{a_i} \cdot \frac{a_{i+2p}}{a_{i+p}} \cdot \frac{a_{i+3p}}{a_{i+2p}} \cdots \frac{a_{i+mp}}{a_{i+(m-1)p}}$$

where $mp \equiv 0 \pmod{n}$.

Put

$$d = (n, p), \quad n = n_1 d.$$

Then from $pr \equiv 0 \pmod n$, (weight condition) we must have $r = r_1 n_1$ (r_1 some integer).

Now if $mp \equiv 0$ the minimum value of m ($\neq 0$) is $m = n_1$. The number of products such as P_m in the set (14.8) is thus $n/n_1 = d$, and we must pick $r/n_1 = r_1$ such P_m products from the set of d. This can be done in $\binom{d}{r_1}$ ways.

This gives the result

Theorem 14.2 The coefficient A of the general two-factor term $a_o^{n-r} a_p^r$ is

$$A = \binom{d}{r_1} = \binom{(n, p)}{\frac{r}{n}(n, p)} \tag{14.9}$$

where $d = (n, p)$, $r_1 = rn/d$.

The coefficient can also be easily obtained by the chain method since the chain-type must be

$$\underbrace{\left[p^{n_1}\right]\left[p^{n_1}\right] \cdots \left[p^{n_1}\right]}_{r_1} \equiv \left[p^{n_1}\right]^{r_1}$$

i.e., r_1 chains all alike.

All the $A_i = 0$ except the one of highest order A_{r_1}. Hence $A = |B|$, and the value of B can be obtained without difficulty.

This also shows that the term $a_o^{n-r} a_p^r$ in the permanent and determined have the same numerical value. And further that a comparison of the two formulas for A, (14.9) and its value as $|B|$ leads to an interesting identity. (See also Ore's paper in this connection).

We shall obtain other instances of such identities whenever a coefficient A is obtained by use of Muir's Theorem, which will express A in terms of the binomial coefficient numbers, and by use of the coefficient B.

Using A as given by (14.9) it is evident that

$$(14.10)\qquad \mathrm{per}\begin{vmatrix} a_o & \cdots & a_p & \cdots & & \\ \cdot & a_o & \cdots & a_p & \cdot\cdot & \\ \cdot & & \cdot & & \cdot\,\cdot & \\ \cdot & & \cdot & & \cdot\cdot & \\ \cdot & & & \cdot & a_p & \\ & & & & \cdot & \\ a_p & \cdot & & & \cdot & \cdot \\ \cdot & & & & & \cdot \\ \cdot\cdots & & a_p & \cdots & \cdot\cdot & a_o \end{vmatrix} = (a_o^{n_1} + a_p^{n_1})^d$$

where the permanent consists of an a_o main diagonal and an a_p off-set diagonal, all other elements being zeros.

(b) <u>Three-factor terms</u>. The general form of such terms is

$$(14.11)\qquad Aa_o^{e_o}a_p^{e_p}a_q^{e_q}\ ,\ (\text{none of } e_o,\ e_p,\ e_q = 0).$$

The simplest case is $\underline{Aa_o^{e_o}a_1^{e_1}a_2^{e_2}}$. Here the weight $w = e_1 + 2e_2 = (e_1 + e_2) + e_2 < n + e_2 < 2n$, i.e. we must have $w = n$, and hence this is a 1-link chain-type term, so that

$$(14.12)\qquad A = \frac{n}{e_1 + e_2}\binom{e_1 + e_2}{e_1}$$

Since $e_o + e_1 + e_2 = n = e_1 + 2e_2$, $e_o = e_2$ and the term can be expressed as $A(a_oa_2)^{e_2}a_1^{e_1}$ with

$$(14.13)\qquad A = \frac{n}{n - e_2}\binom{n - e_2}{e_2}$$

Note that here again $A = |B|$, where B is the coefficient of the corresponding circulant-determinant.

We can now write

$$(14.14)\qquad \left|\begin{matrix} a_0 & a_1 & a_2 & \cdot & \cdot & \cdot \\ \cdot & a_0 & a_1 & a_2 & \cdot & \cdot \\ \cdot & & a_0 & a_1 & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & a_2 \\ a_2 & & & & \cdot & a_1 \\ a_1 & a_2 & \cdots & \cdots & \cdots & a_0 \end{matrix}\right|^{+} = a_0^n + a_2^n + \sum_{x=0}^{x \le \frac{n}{2}} \frac{n}{n-x} \binom{n-x}{x} (a_0 a_2)^x \, a_1^{\,n-2x}$$

(Compare with Ore's expansion of the corresponding determinant).

Consider next the general term (14.11') in which one of p, q is prime to n, say $(p, n) = 1$. Then an equivalent term (same coefficient) can be obtained in which $p = 1$ by use of the multiplier operation (section 1). The term can then be taken as

$$A a_o^{e_o} a_1^{e_1} a_q^{e_q}.$$

Hence, by Muir's Theorem

$A =$ coefficient of $(a_o a_1 \cdots a_{n-1})$ in

$$\frac{a_o a_1 \cdots a_{n-1}}{e_o! \, e_1! \, e_q!} \left|\begin{matrix} 1 \cdots 1 & \frac{a_1}{a_o} \cdots \frac{a_1}{a_o} & \frac{a_q}{a_o} \cdots \frac{a_q}{a_o} \\ \cdot & & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 \cdots 1 & \frac{a_o}{a_{n-1}} \cdots \frac{a_o}{a_{n-1}} & \frac{a_{q-1}}{a_{n-1}} \cdots \frac{a_{q-1}}{a_{n-1}} \\ \longleftarrow e_o \longrightarrow & \longleftarrow e_1 \longrightarrow & \longleftarrow e_q \longrightarrow \end{matrix}\right|^{+}$$

Call the above permanent D. We require the constant term in D. Expand D by its first e_o columns. Each

$$\left|\begin{matrix} 1 & \cdots & 1 \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 1 & \cdots & 1 \end{matrix}\right|^{+}_{e_o} = e_o! \quad ,$$

and its cofactor can be represented by

$$D_{ij\cdots k} = \begin{vmatrix} \frac{a_{i+1}}{a_i} & \cdots & \frac{a_{i+1}}{a_i} & \frac{a_{i+q}}{a_i} & \cdots & \frac{a_{i+q}}{a_i} \\ \frac{a_{j+1}}{a_j} & \cdots & \frac{a_{j+1}}{a_j} & \frac{a_{j+q}}{a_j} & \cdots & \frac{a_{j+q}}{a_j} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \frac{a_{k+1}}{a_k} & \cdots & \frac{a_{k+1}}{a_k} & \frac{a_{k+q}}{a_k} & \cdots & \frac{a_{k+q}}{a_k} \\ \longleftarrow & e_1 & \longrightarrow & \longleftarrow & e_q & \longrightarrow \end{vmatrix}^{+}$$

where $ij \cdots k$ is a selection of $(e_1 + e_q)$ numbers from 0, 1, 2, $\cdots$, n-1.

Expand $D_{ij\cdots k}$ by its first e_1 columns. A typical term is

$$e_1!\; e_q!\; T_{i'j'\cdots k'} U_{i''j''\cdots k''}$$

where

$$T_{i'j'\cdots k'} = \frac{a_{i'+1}}{a_{i'}} \; \frac{a_{j'+1}}{a_{j'}} \cdots \frac{a_{k'+1}}{a_{k'}}, \quad (e_1 \text{ factors})$$

$$U_{i''j''\cdots k''} = \frac{a_{i''+1}}{a_{i''}} \; \frac{a_{j''+1}}{a_{j''}} \cdots \frac{a_{k''+1}}{a_{k''}} \quad (e_q \text{ factors})$$

and $i'j' \cdots k'$ is a selection of e_1 numbers from $ij \cdots k$, and $i''j'' \cdots k''$ is the remainder e_q of elements from $ij \cdots k$.

It follows that

<u>A = the number of products</u> $(T_{i'j'\cdots k'})(U_{i''\cdots k''})$ <u>which = 1</u>.

A may also be represented by means of the following forms equivalent to the above:

$$\begin{array}{ccccc} & & (1) & & \\ 0 & 1 & 2 & \cdots & n-1 \\ 1 & 2 & 3 & \cdots & 0 \\ q & q+1 & q+2 & \cdots & q-1 \end{array} \qquad \begin{array}{ccccc} & & (2) & & \\ \frac{1}{0} & \frac{2}{1} & \frac{3}{2} & \cdots & \frac{0}{n-1} \\ \frac{q}{0} & \frac{q+1}{1} & \frac{q+2}{2} & \cdots & \frac{q-1}{n-1} \end{array}$$

In (2) we must pick e_1 fractions from the top row, say t_1, $t_2, \cdots, t_{e_1}$, and e_q fractions from the bottom row, say $b_1, b_2, \cdots, b_{e_q}$ such that

$$(t_1 \cdots t_{e_1})(b_1 \cdots b_{e_q}) = 1.$$

<u>Also no t and b fraction must be in the same column</u>. The number of such choices is A.

In (1) we must pick e_0 numbers from row 1, e_1 from row 2, and e_q from row q, $(e_0 + e_1 + e_q = n)$, so that there is one and only one number chosen from each column. The number of such choices is A. (In this connection see papers by J. Singer, Partitions and Latin Squares, and "$K_{r,s}^{(n)}$ and $k_{r,s}^{(n)}$")

Illustration: Find A in $Aa_0^4 a_1 a_3^2 (n = 7)$.

$$\begin{array}{ccccccc} 0 & 1 & \underline{2} & \underline{3} & 4 & \underline{5} & \underline{6} \\ \underline{1} & 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & \underline{4} & 5 & 6 & \underline{0} & 1 & 2 \end{array} \qquad \begin{array}{ccccccc} \underline{\frac{1}{0}} & \frac{2}{1} & \frac{3}{2} & \frac{4}{3} & \frac{5}{4} & \frac{6}{5} & \frac{0}{6} \\ \frac{3}{0} & \underline{\frac{4}{1}} & \frac{5}{2} & \frac{6}{3} & \underline{\frac{0}{4}} & \frac{1}{5} & \frac{2}{6} \end{array}$$

By advancing the shown choices cyclically we obtain 7 possible selections, so A = 7. (The (2) method gives a unique choice in the bottom row for each choice in the top row).

In the Summary will be found a variety of coefficients evaluated for certain three-factor terms using the methods just described.

15. Use of Differentiation. Another method which is sometimes useful for evaluating coefficients A is one based on differentiation of the permanent.

Consider first the general circulant-permanent C_n given by (1.1). We can write

$$C_n = \cdots + A a_0^{e_0} a_1^{e_1} \cdots a_{n-1}^{e_{n-1}} + \cdots \qquad (15.1)$$

We first form

$$C_n' = \left(\frac{\partial^{e_0} C_n}{\partial a_0^{e_0}} \right)_{a_0 = 0} ,$$

then

$$C_n'' = \left(\frac{\partial^{e_1} C_n'}{\partial a_1^{e_1}} \right)_{a_1 = 0} , \text{ etc.}$$

The final differentiation will give the value of A. It appears best to pick the a_i with a minimum exponent e_i for the first series of differentiations and work up to the exponents of larger values.

It would not be necessary to carry out the complete set of differentiations as indicated above since the several permanents of relatively small size which are eventually reached could be evaluated directly.

The use of a machine to carry out the various steps outlined above would naturally be desirable.

It should be recalled that $\partial C_n / \partial a_i$ produces n permanents of order (n-1) and all of the same value. (See Theorem [2.5.4] of Part I).

A second way using differentiation is based on Muir's Theorem of the previous section.

If we put

$$D_n = \begin{vmatrix} a_o \cdots a_o & a_1 \cdots a_1 & \cdots & a_{n-1} \cdots a_{n-1} \\ \cdot \quad\quad \cdot & \cdot \quad\quad \cdot & & \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot & \cdot \quad\quad \cdot & & \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot & \cdot \quad\quad \cdot & & \cdot \quad\quad \cdot \\ a_{n-1} \cdots a_{n-1} & a_o \cdots a_o & \cdots & a_{n-2} \cdots a_{n-2} \\ \longleftarrow e_o \longrightarrow & \longleftarrow e_1 \longrightarrow & \cdots & \longleftarrow e_{n-1} \longrightarrow \end{vmatrix}^{+}$$

then to find the coefficient of $(a_o a_1 \cdots a_{n-1})$ we form

$$D'_n = \left(\frac{\partial D_n}{\partial a_o}\right)_{a_o = 0}, \quad D''_n = \left(\frac{\partial D'_n}{\partial a_1}\right)_{a_1 = 0}, \text{ etc.}$$

For example in finding the coefficient of $a_o^4 a_1^2 a_3^2$, $(n = 8)$, we have

$$D'_8 = \left(\frac{\partial D_8}{\partial a_o}\right)_{a_o = 0}$$

$$= 4 \begin{vmatrix} a_1 & a_1 & a_1 & a_2 & a_2 & a_4 & a_4 \\ a_2 & a_2 & a_2 & a_3 & a_3 & a_5 & a_5 \\ a_3 & a_3 & a_3 & a_4 & a_4 & a_6 & a_6 \\ a_4 & a_4 & a_4 & a_5 & a_5 & a_7 & a_7 \\ a_5 & a_5 & a_5 & a_6 & a_6 & 0 & 0 \\ a_6 & a_6 & a_6 & a_7 & a_7 & a_1 & a_1 \\ a_7 & a_7 & a_7 & 0 & 0 & a_2 & a_2 \end{vmatrix}^{+} + 2 \begin{vmatrix} 0 & 0 & 0 & 0 & a_1 & a_3 & a_3 \\ a_1 & a_1 & a_1 & a_1 & a_2 & a_4 & a_4 \\ a_2 & a_2 & a_2 & a_2 & a_3 & a_5 & a_5 \\ a_3 & a_3 & a_3 & a_3 & a_4 & a_6 & a_6 \\ a_4 & a_4 & a_4 & a_4 & a_5 & a_7 & a_7 \\ a_5 & a_5 & a_5 & a_5 & a_6 & 0 & 0 \\ a_6 & a_6 & a_6 & a_6 & a_7 & a_1 & a_1 \end{vmatrix}^{+} +$$

$$+ 2 \begin{vmatrix} 0 & 0 & 0 & 0 & a_1 & a_1 & a_3 \\ a_1 & a_1 & a_1 & a_1 & a_2 & a_2 & a_4 \\ a_2 & a_2 & a_2 & a_2 & a_3 & a_3 & a_5 \\ a_3 & a_3 & a_3 & a_3 & a_4 & a_4 & a_6 \\ a_4 & a_4 & a_4 & a_4 & a_5 & a_5 & a_7 \\ a_6 & a_6 & a_6 & a_6 & a_7 & a_7 & a_1 \\ a_7 & a_7 & a_7 & a_7 & 0 & 0 & a_2 \end{vmatrix}^{+}$$

using column-differentiation.

Here again the use of some machine method would be a necessity for large values of n.

SUMMARY

1. <u>Outline of Procedure.</u> We give here in a rather general form an outline of the various steps to be followed in evaluating coefficients by the method of links and chains.

<u>First step.</u> Let T be the given term. Find an equivalent term to T which has the simplest possible chain decomposition. By equivalent is meant a term obtainable from T by the operation of adding a constant amount to all of its subscripts (Property (1) of section 1). Equivalent terms will of course have the same coefficients. (Note that multiplying all the subscripts of T by the same constant m (prime to n) does not change its chain decomposition).

A simple illustration of this step is given by the term

$$T = a_0 a_{10}^{10} a_{11} a_{12} a_{13} a_{14}, (n=15).$$

This is a 4-link chain term as seen for example by the chain

$$(10,12,10,13)(10,11,10,14)(10,10,10)(10,10,10)$$

Now an equivalent term is

$$T' = a_0^{10} a_1 a_2 a_3 a_4 a_5$$

obtained by adding 5 to the subscripts of T. But T' is a 1-chain term of type $\left[1\ 2\ 3\ 4\ 5\right]$ whose coefficient is immediately found by the formula for this type. To find this coefficient using T would be an involved problem.

In general the simplest equivalent term to T is a t-link chain term where t is a minimum. A good plan is to try the terms with the exponent of a_0 a maximum.

Consider for example

$$T = a_0^2 a_1 a_2^2 a_4 a_5^3 a_6 a_7 a_8 a_9^2 a_{12} \quad , \ (n=15),$$

a rather difficult case. There are 10 equivalent terms containing a zero subscript. Of these, eight are 5-link chain terms and two are 4-link chain terms, these latter being

$$T' = a_0^2 a_2 a_3^3 a_4 a_5 a_6 a_7^2 a_{10} a_{13}^2 a_{14},$$

$$T'' = a_0^3 a_1 a_2 a_3 a_4^2 a_7 a_{10}^2 a_{11} a_{12}^2 a_{14}$$

We would choose T" to work with since it has a larger a_0 exponent.

Second step. Determine all the independent unordered links possible in the chosen equivalent term.

An independent unordered link is an unordered link such that it cannot be decomposed into two or more unordered links. Thus, for the term T" above, [2 4 4 11 12 12] is such an independent unordered link, while [1 2 3 10 14] is a dependent (unordered) link since it can be split into [1 14] [2 3 10].

An independent unordered link is thus one such that every permutation of its elements is a link, while for the dependent type certain of its permutations will be links.

The term T" contains the following such independent links:

[1 14] [1 2 12] [1 3 4 7] [1 2 7 10 10] [2 4 4 11 12 12]

[3 12] [1 3 11] [1 7 10 12] [2 3 4 7 14] [2 4 7 10 10 12]

[4 11] [1 4 10] [2 3 11 14] [2 7 10 12 14]

[2 3 10] [2 4 10 14] [3 7 10 11 14]

[4 4 7] [2 4 12 12] [4 7 10 10 14]

[7 11 12] [2 7 10 11]

[3 7 10 10]

[4 4 10 12]

[7 12 12 14]

[10 10 11 14]

[10 11 12 12]

<u>Third step.</u> Use these independent links to build up all possible unordered chains. This gives what was previously called the <u>chain-type</u> of the term.

For T" a partial list of such chains is

[2 4 7 10 10 12] [1 14] [3 12] [4 11] ,

[2 7 10 12 14] [1 4 10] [3 12] [4 11] ,

[10 11 12 12] [2 3 10] [4 4 7] [1 14] ,

[2 3 4 7 14] [10 11 12 12] [1 4 10] ,

[1 7 10 12] [2 3 11 14] [4 4 10 12]

(No 3-link or 1-link chains of this nature are possible for T").

From this list of <u>independent</u> chains we then form all unordered <u>dependent</u> chains, i.e. chains whose links may be dependent.

Thus, from the partial list above we have

[2 3 4 4 7 10 10 11 12 12] [1 14],
[2 4 4 7 10 10 11 12] [1 3 12 14],
[1 3 4 11 12 14] [2 4 7 10 10 12],
[2 7 10 12 14] [3 4 11 12] [1 4 10]

as examples of such dependent unordered chains.

<u>Fourth step.</u> Divide all the unordered chains into two sets, those of an even number of links and those of an odd number of links. Choose one set or the other, and then evaluate S_1 or S_2 , i.e.

$$A_1 + A_3 + A_5 + \ldots \text{ or } A_2 + A_4 + A_6 + \ldots$$

To find the contribution of any particular unordered chain it is necessary to know all of its link-permutations. In the case of an independent link all permutations of its elements are allowable. Thus in the chain

[2 4 7 10 10 12] [1 3 4 11 12 14]

the first link is independent and gives rise to $5!/2!$ link-permutations (keeping, e.g., the 2 fixed). The second link is dependent. For such a link of short length its link-permutations can be obtained by selecting them from all possible permutations. Here we would have $5!$ such permutations to select from (one element fixed). However, for longer links this is not practical by hand. The following method appears convenient for machine use.

Suppose we have the dependent link $[x_1x_2\dots x_m]$, to find all its link-permutations. Form all the length-two sequences $x_1x_2, x_1x_3, \dots, x_1x_m$ (exclude x_1x_1' if $x_1+x_1' \equiv 0$). For each of these form all allowable length-three sequences $x_1x_2x_3, x_1x_2x_4, \dots, x_1x_mx_{m-1}$. A sequence $x_1x_jx_k$ is excluded if $x_1 + x_j + x_k \equiv 0$ or $x_j + x_k \equiv 0$. Continue in this way till all allowable length-m sequences are obtained. These will be the required link-permutations. (At the $(r-1)^{st}$ step a length-r sequence $x_1x_2x_3\dots x_r$ is excluded if

$x_i + x_{i+1} + \dots + x_r \equiv 0$ for any $i \geq 1$).

For every particular ordered chain (of t-links) (i.e., a chain each of whose links is a definite link-permutation) we must find the value of $N_{a_1a_2\dots a_{t-1}}(x_t)$. (See (6.8)). This is used to evaluate A_t. A method for this is described in section 8.

After finding S_1 (or S_2) we find the coefficient B of the circulant-determinant, (Ore's formula) and then evaluate A with its aid.

Note that the first three steps outlined above are essentially the ones used to obtain the value of B.

Various types of simplifications of the above procedures will no doubt be suggested on further study.

2. List of general formulas.

The following formulas are those derived in the previous sections plus some additional ones.

1-chain terms

Chain-type:

$$[G_1] = [p_1^{m_1} p_2^{m_2} \dots p_g^{m_g}]$$

Coefficient:

$$A = \frac{n\,(L-1)\,!}{m_1!\; m_2!\; \ldots\; m_g!} \quad , \qquad L = m_1 + m_2 + \ldots + m_g$$

2- chain terms

Chain-type

$$[G_1]\,[G_2] = \left[p_1^{m_1} \ldots p_g^{m_g}\right]\left[q_1^{n_1} \ldots q_h^{n_h}\right], \quad (p_i \neq q_j, \text{ all } i,j).$$

$$A = \frac{1}{\pi_{G_1}\,\pi_{G_2}} \left\{ (L_1 - 1)!\,(L_2 - 1)!\,n^2 + \left[(L-1)! - 2L_1!\,L_2!\right] n \right\}$$

$$\pi_{G_1} = m_1!\,m_2!\,\ldots\,m_g!\,, \qquad \pi_{G_2} = n_1!\,\ldots n_h!$$

$$L_1 = m_1 + \ldots + m_g, \qquad L_2 = n_1 + \ldots + n_h\,, \qquad L = L_1 + L_2\,.$$

Chain-Type

$$\left[r^{u_1} G_1\right]\left[r^{v_1} G_2\right]$$

It will be assumed hereafter that G_1, G_2 are defined as above with $p_i \neq q_j$ for all i,j.

$$A = \frac{1}{\pi_{G_1}\pi_{G_2}} \left\{ \frac{(L_1-1)!\,(L_2-1)!}{u_1!\;v_1!}\,n^2 + \left[\frac{(L-1)!}{(u_1+v_1)!} - 2\,\frac{L_1!\;L_2!}{u_1!\;v_1!} + 2\,\frac{(L_1-1)!\;(L_2-1)!}{(u_1-1)!\;(v_1-1)!}\right] n \right\}$$

$$L_1 = u_1 + \sum m_i, \qquad L_2 = v_1 + \sum n_i\,, \qquad L = L_1 + L_2$$

Exceptional case: $m_i = 0, n_i = 0$, (indicated by $G_1 = G_2 = 0$),

$u_1 = v_1$, or type $\left[r^{u_1}\right]^2$;

$$A = \frac{n}{2u_1^2}\,(n - u_1)$$

Chain-type

$$\left[r^{u_1} s^{u_2} G_1\right]\left[r^{v_1} s^{v_2} G_2\right]$$

$$A = \frac{1}{\pi_{G_1}\pi_{G_2}}\left\{\frac{(L_1-1)!\,(L_2-1)!}{u_1!\,u_2!\,v_1!\,v_2!}\,n^2 + \left[\frac{(L-1)!}{(u_1+v_1)!\,(u_2+v_2)!}\right.\right.$$

$$\left.\left. + 2\sum_{i_1+i_2=0} I(i_1,i_2)M_{i_1 i_2}\right]n\right\}$$

$$M_{i_1 i_2} = \frac{\left(L_1-(i_1+i_2)\right)!\,\left(L_2-(i_1+i_2)\right)!}{(u_1-i_1)!\,(u_2-i_2)!\,(v_1-i_1)!\,(v_2-i_2)!}, \quad \text{(all } i_1, i_2),$$

$$I(i_1,i_2) = 2\,\frac{i_1+i_2-1}{i_1 i_2}\binom{i_1+i_2-2}{i_2-1}\binom{i_1+i_2-2}{i_1-1}, \quad \left(\begin{matrix} i_1+i_2 \geq 2, \\ i_1 i_2 \neq 0 \end{matrix}\right),$$

$$I(0,0) = -1, \quad I(1,0) = I(0,1) = 1,$$

$$L_1 = u_1+u_2+\sum m_i, \quad L_2 = v_1+v_2+\sum n_i, \quad L = L_1+L_2$$

Exceptional case; chain-type $\left[r^{u_1}s^{u_2}\right]^2$

$$A = \frac{1}{2(u_1+u_2)^2}\binom{u_1+u_2}{u_1}^2 n^2 - \frac{1}{2(u_1+u_2)}\binom{2u_1+2u_2}{2u_1} n$$

Chain-type

$$\left[r_1^{u_1} r_2^{u_2} \ldots r_d^{u_d} G_1\right]\left[r_1^{v_1} r_2^{v_2} \ldots r_d^{v_d} G_2\right]$$

(the general 2-chain type)

$$A = \frac{1}{\pi_{G_1}\pi_{G_2}}\left\{\frac{(L_1-1)!\,(L_2-1)!}{u_1!\ldots u_d!\,v_1!\ldots v_d!}\,n^2\right.$$

$$\left. + \left[\frac{(L-1)!}{(u_1+v_1)!\ldots(u_d+v_d)!} + 2\sum I(i_1,i_2,\ldots,i_d;r_\alpha,r_\beta)M(i_1,\ldots,i_d)\right] n\right\}$$

$$L_1 = \sum u_i + \sum m_i, \quad L_2 = \sum v_i + \sum n_i$$

$$M(i_1,\ldots,i_d) = \frac{(L_1-(i_1+\ldots+i_d))!\,(L_2-(i_1+\ldots+i_d))!}{(u_1-i_1)!\ldots(u_d-i_d)!\,(v_1-i_1)!\ldots(v_d-i_d)!}$$

See (11.16), (11.17) for definition of I function.

Exceptional case:

$$\left[r_1^{u_1} r_2^{u_2} \cdots r_d^{u_d}\right]\left[r_1^{u_1} r_2^{u_2} \cdots r_d^{u_d}\right]$$

$$A = \frac{1}{2}\left[\frac{(L-1)!}{u_1!\cdots u_d!}\right]^2 n^2 + \left[\frac{(2L-1)!}{(2u_1)!\cdots(2u_d)!} + \sum I(i_1,\ldots,i_d;r_\alpha,r_\beta)M(i_1,\ldots,i_d) - \sum\frac{i_k}{u_k} I(i_1,\ldots,i_d;r_\alpha,r_\beta)M(i_1,\ldots,i_d)\right] n$$

where k is fixed but may equal 1,2,..., or d.

Special cases of general 2-chain type

Type $\left[r^{u_1} s^{v_1} G_1\right]\left[r\ s\ G_2\right]$

$$A = \frac{n}{u_1!v_1!\pi_{G_1}\pi_{G_2}}\left[(L_1-1)!(L_2-1)!n + \frac{(L-1)!}{(u_1+1)(v_1+1)} - 2L_1!\,L_2! + 2(u_1+v_1)(L_1-1)!(L_2-1)! + 4u_1v_1(L_1-2)!(L_2-2)!\right]$$

Type $\left[r^{u_1} s^{v_1} G_1\right]\left[r^2 s\ G_2\right]$

$$A = \frac{n}{u_1!v_1!\pi_{G_1}\pi_{G_2}}\left[\frac{1}{2}(L_1-1)!(L_2-1)!\,n + \frac{(L-1)!}{(u_1+2)(u_1+1)(v_1+1)} - L_1!\,L_2! + (2u_1+v_1)(L_1-1)!(L_2-1)! + 4u_1v_1(L_1-2)!(L_2-2)! + 4(u_1+1)u_1v_1(L_1-3)!(L_2-3)!\right]$$

Type $\left[r^{u_1} s^{u_2} G_1\right]\left[r^2 s^2 G_2\right]$

$$A = \frac{(L_1-1)!(L_2-1)!}{4u_1!u_2!\pi_{G_1}\pi_{G_2}} n^2 + \frac{n}{\pi_{G_1}\pi_{G_2}}\left[\frac{(L-1)!}{(u_1+2)!(u_2+2)!} - \frac{L_1!\,L_2!}{2u_1!u_2!} + \frac{(u_1+u_2)(L_1-1)!(L_2-1)!}{u_1!\,u_2!} + \frac{4(L_1-2)!\,(L_2-2)!}{(u_1-1)!\,(u_2-1)!} + 4\frac{(u_1+u_2-2)}{(u_1-1)!(u_2-1)!}(L_1-3)!(L_2-3)! + 12\frac{(L_1-4)!(L_2-4)!}{(u_1-2)!(u_2-2)!}\right]$$

<u>Type</u> $\left[r^{u_1} s^{u_2} G_1\right] \quad \left[r^3 s^2 G_2\right]$

$$A = \frac{(L_1-1)!(L_2-1)!}{12u_1!u_2!\pi_{G_1}\pi_{G_2}}\, n^2 + \frac{n}{\pi_{G_1}\pi_{G_2}} \left[\frac{(L-1)!}{(u_1+3)!(u_2+2)!} - \frac{L_1!\ L_2!}{6u_1!u_2!} \right.$$

$$+ \frac{(3u_1+2u_2)}{6u_1!u_2!}(L_1-1)!(L_2-1)! + \frac{2(L_1-2)!(L_2-2)!}{(u_1-1)!(u_2-1)!}$$

$$+ \frac{(4u_1+2u_2-6)}{(u_1-1)!(u_2-1)!}(L_1-3)!(L_2-3)! + \frac{(4u_1+12u_2-20)}{(u_1-2)!(u_2-1)!}(L_1-4)!(L_2-4)!$$

$$\left. + 24\frac{(L_1-5)!\ (L_2-5)!}{(u_1-3)!(u_2-2)!} \right]$$

<u>Type</u> $\left[r^{u_1} s^{u_2} G_1\right] \quad \left[r^3 s^3 G_2\right]$

$$A = \frac{(L_1-1)!(L_2-1)!}{36\ u_1!u_2!\pi_{G_1}\pi_{G_2}}\, n^2 + \frac{n}{\pi_{G_1}\pi_{G_2}} \left[\frac{(L-1)!}{(u_1+3)!(u_2+3)!} - \frac{L_1!L_2!}{18u_1!u_2!} \right.$$

$$+ \frac{u_1+u_2}{6u_1!u_2!}(L_1-1)!(L_2-1)! + \frac{(L_1-2)!(L_2-2)!}{(u_1-1)!(u_2-1)!} + \frac{2(u_1+u_2-2)(L_1-3)!(L_2-3)!}{(u_1-1)!(u_2-1)!}$$

$$+ 2\Big((u_1-1)(u_1-2)+(u_2-1)(u_2-2)+6(u_1-1)(u_2-1)\Big)\frac{(L_1-4)!(L_2-4)!}{(u_1-1)!(u_2-1)!}$$

$$\left. + \frac{24(u_1+u_2-4)}{(u_1-2)!(u_2-2)!}(L_1-5)!(L_2-5)! + 80\frac{(L_1-6)!(L_2-6)!}{(u_1-3)!(u_2-3)!} \right]$$

Type $\left[r^{u_1} s^{u_2} G_1\right] \left[r^{v_1} s \; G_2\right]$

$$A = \frac{(L_1-1)!(L_2-1)!}{u_1!v_1!u_2!\pi_{G_1}\pi_{G_2}} n^2 + \frac{n}{\pi_{G_1}\pi_{G_2}}\left[\frac{(L-1)!}{(u_1+v_1)!(u_2+1)!} - 2\frac{L_1!L_2!}{u_1!v_1!u_2!}\right.$$

$$\left. + \frac{2(u_1v_1+u_2)}{u_1!v_1!u_2!}(L_1-1)!(L_2-1)! + \frac{4}{(u_2-1)!}\sum_{i=1}\frac{(L_1-i-1)!(L_2-i-1)!}{(u_1-i)!(v_1-i)!}\right]$$

Type $\left[r_1r_2r_3\right] \left[r_1r_2r_3\right]$

$$A = 2n^2 - 11n$$

Type $\left[r_1^2r_2r_3\right]\left[r_1^2r_2r_3\right]$

$$A = \frac{1}{2}(9n^2 - 73n)$$

Type $\left[r_1^{u_1} r_2^{u_2} r_3^{u_3} G_1\right] \left[r_1r_2r_3G_2\right]$

$$A = \frac{n}{u_1!u_2!u_3!\pi_{G_1}\pi_{G_2}}\left[(L_1-1)!(L_2-1)!\,n + \frac{(L-1)!}{(u_1+1)(u_2+1)(u_3+1)} - 2L_1!L_2!\right.$$

$$+ 2(u_1+u_2+u_3)(L_1-1)!(L_2-1)! + 4(u_1u_2+u_1u_3+u_2u_3)(L_1-2)!(L_2-2)!$$

$$\left. + 36\,u_1u_2u_3(L_1-3)!(L_2-3)!\right]$$

Type $\left[r_1^{u_1} r_2^{u_2} r_3^{u_3} G_1\right] \left[r_1^2 r_2r_3G_2\right]$

$$A = \frac{n}{u_1!u_2!u_3!\pi_{G_1}\pi_{G_2}}\left[\frac{1}{2}(L_1-1)!(L_2-1)!\,n + \frac{(L-1)!}{(u_1+2)(u_1+1)(u_2+1)(u_3+1)}\right.$$

$$- L_1!L_2! + (2u_1+u_2+u_3)(L_1-1)!(L_2-1)!$$

$$+ 2(2u_1u_2+2u_1u_3+u_2u_3)(L_1-2)!(L_2-2)!$$

$$\left. + 4\Big(9u_1u_2u_3+u_1(u_1-1)(u_2+u_3)\Big)(L_1-3)!(L_2-3)! + 80\,u_1u_2u_3(u_1-1)\right]$$

<u>Type</u> $\left[r_1^{u_1} r_2^{u_2} r_3^{u_3} G_1\right] \left[r_1^2 r_2^2 r_3 G_2\right]$

$$A = \frac{(L_1-1)!(L_2-1)!}{4u_1!u_2!u_3!\pi_{G_1}\pi_{G_2}} n^2 + \frac{n}{\pi_{G_1}\pi_{G_2}}\left[\frac{(L-1)!}{(u_1+2)!(u_2+2)!(u_3+1)!} - \frac{L_1!L_2!}{2u_1!u_2!u_3!}\right.$$

$$+ \frac{(2u_1+2u_2+u_3)}{2u_1!u_2!u_3!}(L_1-1)!(L_2-1)! + \frac{2(2u_1u_2+u_1u_3+u_2u_3)}{u_1!u_2!u_3!}(L_1-2)!(L_2-2)!$$

$$+\left(36u_1u_2u_3+2u_1u_3(u_1-1)+2u_2u_3(u_2-1)+4u_1u_2(u_1-1)+4u_1u_2(u_2-1)\right)\frac{(L_1-3)!(L_2-3)!}{u_1!u_2!u_3!}$$

$$+\left(88u_1u_2u_3(u_1+u_2-2)+12u_1u_2(u_1-1)(u_2-1)\right)\frac{(L_1-4)!(L_2-4)!}{u_1!u_2!u_3!}$$

$$\left. + 516\frac{(L_1-5)!(L_2-5)!}{(u_1-2)!(u_2-1)!(u_3-1)!}\right]$$

<u>Type</u> $\left[r_1^{u_1} r_2^{u_2} r_3^{u_3} G_1\right] \left[r_1^2 r_2^2 r_3^2 G_2\right]$

$$A = \frac{n}{(2!)^3 u_1!u_2!u_3!\pi_{G_1}\pi_{G_2}}\left\{(L_1-1)!(L_2-1)!\, n + \right.$$

$$+ \frac{8(L-1)!}{(u_1+2)(u_1+1)(u_2+2)(u_2+1)(u_3+2)(u_3+1)} - 2L_1!L_2!$$

$$+ 4(u_1+u_2+u_3)(L_1-1)!L_2-1)! + 16(u_1u_2+u_1u_3+u_2u_3)(L_1-2)!(L_2-2)!$$

$$+\left\{16\left(u_1(u_1-1)(u_2+u_3)+u_2(u_2-1)(u_3+u_1)+u_3(u_3-1)(u_1+u_2)\right)\right.$$

$$\left.+ 288\, u_1u_2u_3\right\}(L_1-3)!(L_2-3)!$$

$$+ 8\left\{88u_1u_2u_3(u_1+u_2+u_3-3) + 3\left(u_1u_2(u_1-1)(u_2-1) + u_1u_3(u_1-1)(u_3-1)\right.\right.$$

$$\left.\left.+ u_2u_3(u_2-1)(u_3-1)\right)\right\} (L_1-4)!(L_2-4)!$$

$$+ 8(516)u_1u_2u_3\left\{(u_1-1)(u_2-1)+(u_1-1)(u_3-1)+(u_2-1)(u_3-1)\right\}(L_1-5)!(L_2-5)!$$

$$\left. + 8(4{,}596)\, u_1u_2u_3(u_1-1)(u_2-1)(u_3-1)(L_1-6)!(L_2-6)!\right]$$

Type $\left[r_1^{u_1} r_2^{u_2} r_3^{u_3} r_4^{u_4} G_1\right]\left[r_1 r_2 r_3 r_4 G_2\right]$

$$A = \frac{n}{u_1!u_2!u_3!u_4!\pi_{G_1}\pi_{G_2}}\left\{(L_1-1)!(L_2-1)!n + \frac{(L-1)!}{(u_1+1)(u_2+1)(u_3+1)(u_4+1)}\right.$$

$$- 2L_1!L_2! + 2(u_1+u_2+u_3+u_4)(L_1-1)!(L_2-1)!$$

$$+ 4(u_1u_2+u_1u_3+u_1u_4+u_2u_3+u_2u_4+u_3u_4)(L_1-2)!(L_2-2)!$$

$$+ 36(u_1u_2u_3+ u_1u_2u_4+u_1u_3u_4+u_2u_3u_4)(L_1-3)!(L_2-3)!$$

$$\left. + 624\, u_1u_2u_3u_4(L_1-4)!(L_2-4)!\right\}$$

Type $\left[r_1^{u_1} r_2^{u_2} r_3^{u_3} r_4^{u_4} G_1\right]\left[r_1^2 r_2 r_3 r_4 G_2\right]$

$$A = \frac{n}{u_1!\ldots u_4!\pi_{G_1}\pi_{G_2}}\left[\frac{1}{2}(L_1-1)!(L_2-1)!n + \frac{(L-1)!}{(u_1+2)(u_1+1)(u_2+1)(u_3+1)(u_4+1)}\right.$$

$$- L_1!L_2! + (2u_1+u_2+u_3+u_4)(L_1-1)!(L_2-1)!$$

$$+ 2(2u_1u_2+2u_1u_3+2u_1u_4+u_2u_3+u_2u_4+u_3u_4)(L_1-2)!(L_2-2)!$$

$$+\left\{4u_1(u_1-1)(u_2+u_3+u_4) + 18(2u_1u_2u_3+2u_1u_2u_4+2u_1u_3u_4+u_2u_3u_4)\right\}(L_1-3)!(L_2-3)!$$

$$+\left\{100u_1(u_1-1)(u_2u_4+u_3u_4+u_2u_3) + 624\, u_1u_2u_3u_4\right\}(L_1-4)!(L_2-4)!$$

$$\left. + 3{,}204\, u_1(u_1-1)\, u_2u_3u_4(L_1-5)!\,(L_2-5)!\right]$$

In the following formulas

$$G_3 = t_1^{\alpha_1}t_2^{\alpha_2}\ldots t_i^{\alpha_i}, \quad G_4 = w_1^{\beta_1}w_2^{\beta_2}\ldots w_j^{\beta_j},$$

$$G_5 = x_1^{\gamma_1}x_2^{\gamma_2}\ldots x_k^{\gamma_k}, \quad G_6 = y_1^{\delta_1}y_2^{\delta_2}\ldots y_f^{\delta_f},$$

and no two G's have a common element.

Also

$$g_1 = m_1+\ldots+m_g, \quad g_2 = n_1+\ldots+n_h, \quad g_3 = \alpha_1+\ldots+\alpha_i,$$

$$g_4 = \beta_1+\ldots+\beta_j, \quad g_5 = \gamma_1+\ldots+\gamma_k, \quad g_6 = \delta_1+\ldots+\delta_f$$

<u>Type</u> $\left\{ \left[G_1G_2\right]\left[G_3G_4\right],\ \left[G_1G_3\right]\left[G_2G_4\right] \right\}$

$$A = \frac{1}{\pi_G}\left\{ \left[(L_1-1)!(L_2-1)! + (M_1-1)!(M_2-1)!\right] n^2 + \left[(L-1)! - 2\, L_1!L_2! - 2\, M_1!M_2! + 4g_1!g_2!g_3!g_4!\right] n \right\},$$

$L_1 = g_1 + g_2$, $L_2 = g_3 + g_4$, $M_1 = g_1 + g_3$, $M_2 = g_2 + g_4$, $L = L_1 + L_2$,

$\pi_G = \pi_{G_1}\pi_{G_2}\pi_{G_3}\pi_{G_4}$

<u>Type</u> $\left\{ \left[G_1G_2G_3\right]\left[G_4G_5\right],\ \left[G_1G_3G_4\right]\left[G_2G_5\right],\ \left[G_2G_3G_4\right]\left[G_1G_5\right] \right\}$

$$A = \frac{1}{\pi_G}\left\{ \left[(L_1-1)!\,(L_2-1)! + (M_1-1)!(M_2-1)! + (N_1-1)!(N_2-1)!\right] n^2 + \left[(L-1)! - 2(L_1!L_2! + M_1!M_2! + N_1!N_2!) + 2E\right] n \right\}$$

$$E = g_1!g_2!\dots g_5!\left[\binom{g_1+g_3}{g_1} + \binom{g_2+g_3}{g_2} + \binom{g_4+g_3}{g_4}\right],$$

$L_1 = g_1 + g_2 + g_3$, $L_2 = g_4 + g_5$, $M_1 = g_1 + g_3 + g_4$, $M_2 = g_4 + g_5$,

$N_1 = g_2 + g_3 + g_4$, $N_2 = g_1 + g_5$, $L = L_1 + L_2$

π_G as above with π_{G_5} added.

<u>Type</u> $\left\{ \left[r^{b_1+b_2}G_1G_2\right]\left[r^{b_3+b_4}G_3G_4\right],\ \left[r^{b_1+b_3}G_1G_3\right]\left[r^{b_2+b_4}G_2G_4\right] \right\}$

$$A = \frac{1}{\pi_G}\left[\frac{(L_1-1)!(L_2-1)!}{(b_1+b_2)!(b_3+b_4)!} + \frac{(M_1-1)!(M_2-1)!}{(b_1+b_3)!(b_2+b_4)!}\right] n^2$$

$$+ \frac{1}{\pi_G}\left[\frac{(L-1)!}{b!} - \frac{2\, L_1!L_2!}{(b_1+b_2)!(b_3+b_4)!} - \frac{2\, M_1!M_2!}{(b_1+b_3)!(b_2+b_4)!} + \frac{2(L_1-1)!(L_2-1)!}{(b_1+b_2-1)!(b_3+b_4-1)!} + \frac{2(M_1-1)!(M_2-1)!}{(b_1+b_3-1)!(b_2+b_4-1)!} + 4\sum_{x=0} E(x)\right] n$$

$$E(x) = g_1!\dots g_4!\left[\binom{x+g_1}{g_1}\binom{x+b_4-b_1+g_4}{g_4} - 2\binom{x+g_1-1}{g_1}\binom{x+b_4-b_1+g_4-1}{g_4} + \binom{x+g_1-2}{g_1}\binom{x+b_4-b_1+g_4-2}{g_4}\right]\binom{b_1+b_2+g_2-x}{g_2}\binom{b_1+b_3+g_3-x}{g_3},$$

$L_1 = b_1+b_2+g_1+g_2$, $\quad L_2 = b_3+b_4+g_3+g_4$, $\quad M_1 = b_1+b_3+g_1+g_3$,

$M_2 = b_2+b_4+g_2+g_4$, $\quad L = L_1+L_2$, $\quad b = b_1+b_2+b_3+b_4$

<u>3-chain terms</u>

<u>Type</u> $\quad [G_1] \quad [G_2] \quad [G_3]$

$$A = \frac{n}{\pi_G}\Big[(L_1-1)!(L_2-1)!(L_3-1)!\,n^2 + \Big\{(L_1-1)!(L_2+L_3-1)!+(L_2-1)!(L_3+L_1-1)!$$

$$+ (L_3-1)!(L_1+L_2-1)!-2(L_1-1)!(L_2-1)!(L_3-1)!(L_2L_3+L_3L_1+L_1L_2)\Big\}\,n$$

$$+(L-1)!-2\Big\{(L_1+L_2)!L_3!+(L_2+L_3)!L_1!+(L_1+L_3)!L_2!\Big\}+4(L_1+L_2+L_3)L_1!L_2!L_3!\Big]$$

$L_1 = g_1$, $\quad L_2 = g_2$, $\quad L_3 = g_3$, $\quad \pi_G = \pi_{G_1}\pi_{G_2}\pi_{G_3}$

<u>Type</u> $\quad [r^{u_1}G_1] \quad [r^{u_2}G_2] \quad [r^{u_3}G_3]$

$$A = \frac{1}{\pi_G}\Big\{\frac{(L_1-1)!(L_2-1)!(L_3-1)!}{u_1!\,u_2!\,u_3!}\,n^3$$

$$+\sum_{i=1}^{3}\left[\frac{(L-L_i-1)!(L_i-1)!}{(u-u_i)!\,u_i!}-2\frac{L_1!L_2!L_3!}{u_1!u_2!u_3!}\frac{1}{L_i}+\frac{2(L_1-1)!(L_2-1)!(L_3-1)!}{(u_1-1)!(u_2-1)!(u_3-1)!}\frac{1}{u_i}\right]n^2$$

$$+\Big\{\frac{(L-1)!}{u!}+2\sum_i\frac{(L-L_i-1)!(L_i-1)!}{(u-u_i)!\,u_i!}\Big[(u-u_i)u_i-(L-L_i)L_i\Big]$$

$$-4\frac{L_1!L_2!L_3!}{u_1!u_2!u_3!}\sum_i\frac{u_i(u-u_i)}{L_i}$$

$$+4\frac{(L_1-1)!(L_2-1)!(L_3-1)!}{(u_1-1)!(u_2-1)!(u_3-1)!}\left(\frac{L_1L_2L_3L}{u_1u_2u_3}+u\right)n\Big\}$$

Here $u = u_1+u_2+u_3$, $\quad L_i = u_i+g_i$

Two Exceptional cases;

<u>Type</u> $\quad [r^{u_1}] \quad [r^{u_1}] \quad [r^{u_3}G_3]$

$$A = \frac{1}{\pi_{G_3}}\Big\{\frac{(L_3-1)!}{2u_1^2u_3!}\,n^3+\frac{1}{u_1}\left[\frac{(L_3+u_1-1)!}{(u_1+u_3)!}-2\frac{(L_3-1)!}{u_3!}(L_3-u_3)-\frac{(L_3-1)!}{2u_3!}\right]n^2$$

$$+\left[\frac{(L-1)!}{(2u_1+u_3)!}-2(L_3-u_3)\left(\frac{(L_3+u_3-1)!}{(u_1+u_3)!}-\frac{(L_3-1)!}{u_3!}(L_3-u_3)\right]n\Big\}$$

<u>Type</u> $\left[r^{u_1} \right]^3$

$$A = \frac{n}{3!u_1^{\,3}} \quad (n-u_1)(n-2u_1)$$

<u>Type</u> $\left\{ [G_1G_2][G_3G_4][G_5G_6], [G_1G_3G_5][G_2G_4G_6] \right\}$

$$A = \frac{1}{\pi_G}\Big\{ (L_1-1)!(L_2-1)!(L_3-1)!\, n^3$$

$$+\left[\sum_{i=1}^{3} (L_i-1)!(L-L_i-1)! - 2\sum (L_i-1)!L_j!L_k! + (M_1-1)!(M_2-1)! \right] n^2$$

$$+\Big[4(L_1+L_2+L_3)L_1!L_2!L_3! + (L-1)! - 2M_1!M_2! - 2\sum L_i!(L-L_i)!$$

$$+ 4M_1!M_2!\,(P) + 2\left(2L-M_1M_2+(g_1g_2+g_3g_4+g_5g_6)-14\right) g_1!g_2!\ldots g_6! \Big]\, n \Big\}$$

$$L_1 = g_1+g_2, \quad L_2 = g_3+g_4, \quad L_3 = g_5+g_6, \quad M_1 = g_1+g_3+g_5, \quad M_2 = g_2+g_4+g_6$$

$$L = L_1 + L_2 + L_3, \quad \pi_G = \pi_{G_1} \ldots \pi_{G_6}$$

$$P = \frac{1}{\binom{M_1}{g_1}\binom{M_2}{g_2}} + \frac{1}{\binom{M_1}{g_3}\binom{M_2}{g_4}} + \frac{1}{\binom{M_1}{g_5}\binom{M_2}{g_6}}$$

4 - chain term

Type $\quad [G_1]\ [G_2]\ [G_3]\ [G_4]$

$$A = \frac{1}{\pi_G} \Big\{ (L_1-1)!(L_2-1)!(L_3-1)!(L_4-1)!\, n^4$$

$$+ \Big[\sum_{(4)}^{\circ} (L_1-1)!(L_2-1)!(L_3+L_4-1)! + \sum_{(2)}^{\circ} (L_1-1)!(L_3-1)!(L_2+L_4-1)!$$

$$- 2\sum_{(4)}^{\circ} (L_1-1)!(L_2-1)!\, L_3!L_4! - 2\sum_{(2)}^{\circ} (L_1-1)!(L_3-1)!\, L_2!L_4! \Big]\, n^3$$

$$+ \Big[(L_1+L_2-1)!(L_3+L_4-1)! + (L_1+L_3-1)!(L_2+L_4-1)! + (L_1+L_4-1)!(L_2+L_3-1)!$$

$$+ \sum_{i=1}^{4} (L_i-1)!(L-L_i-1)! -$$

$$- 2\sum_{(4)}^{\circ} (L_1-1)!(L_2-1)! \Big((L_3+L_4-1)! - L_3!L_4! \Big) \Big(L_1L_2 + (L_1+L_2)(L_3+L_4) \Big)$$

$$-2\sum_{(2)}^{\circ} (L_1-1)!(L_3-1)! \Big((L_2+L_4-1)! - L_2!L_4! \Big) \Big(L_1L_3 + (L_1+L_3)(L_2+L_4) \Big) \Big]\, n^2$$

$$+ \Big[(L-1)! - 2\sum_{1}^{4} L_i!(L-L_i-1)! + 2\sum_{(4)}^{\circ} L_1!L_2! \Big((L_3+L_4-1)! - L_3!L_4! \Big)(L_3+L_4)(L-1)$$

$$+ 2\sum_{(2)}^{\circ} L_1!L_3! \Big((L_2+L_4-1)! - L_2!L_4! \Big)(L_2+L_4)(L-1) + 2(L+1)\sum_{(4)}^{\circ} (L_1+L_2)!L_3!L_4!$$

$$+ 2(L+1)\sum_{(2)}^{\circ} (L_1+L_3)!L_2!L_4!$$

$$- 2\Big((L_1+L_2)!(L_3+L_4)! + (L_1+L_3)!(L_2+L_4)! + (L_1+L_4)!(L_2+L_3)! \Big)$$

$$- 2(L+1)(L+2)L_1!L_2!L_3!L_4! \Big]\, n$$

$L_i = g_i$, $L = \sum L_i$. The $\sum^{\circ}$ means a cyclic sum obtained by advancing the subscripts on the L's one unit at a time to include only distinct terms so derived. Number of such distinct terms is indicated in parentheses under the $\sum^{\circ}$ sign.

Two-factor and three-factor terms

Type $a_o^{n-r} \; a_p^{r}$ (general 2-factor term)

$$A = \binom{d}{r_1}, \quad \text{where } d = (n,p), \quad r_1 = rd/n$$

Three-factor terms

Type $\left[r^u s^v\right]^3$

$$A = \frac{1}{(3u)!(3v)!}\left\{\frac{1}{3!}\binom{3u}{u}\binom{3v}{v}\binom{2u}{u}\binom{2v}{v}\left[(L_1-1)!\right]^3 n^3 - \binom{3u}{u}\binom{3v}{v}(2L_1-1)!\,(L_1-1)!\,n^2 + (L-1)!\,n\right\}$$

$L_1 = u+v, \quad L = 3(u+v)$

Type $\left[r^u s^v\right]^4$

$$A = \frac{n}{(4u)!(4v)!}\left\{\frac{1}{4!}\binom{4u}{u}\binom{4v}{v}\binom{3u}{u}\binom{3v}{v}\binom{2u}{u}\binom{2v}{v}\left((L_1-1)!\right)^4 n^3\right.$$

$$-\frac{1}{3!}\left[3\binom{4u}{u}\binom{4v}{v}\binom{3u}{u}\binom{3v}{v}\left((L_1-1)!\right)^2 (2L_1-1)!\right] n^2$$

$$+\frac{1}{2!}\left[\binom{4u}{u}\binom{4v}{v}(L_1-1)!(3L_1-1)! + \binom{4u}{2u}\binom{4v}{2v}\left((2L_1-1)!\right)^2\right.$$

$$\left.\left.+\binom{4u}{3u}\binom{4v}{v}(L_1-1)!(3L_1-1)!\right] n - (L-1)!\right\}$$

$L_1 = u+v, \quad L = 4L_1$

 90.

<u>Type</u> $\left[r^{u}s^{v}\right]^{5}$

$$A = n\left\{\frac{\left((L_1-1)!\right)^5}{5!\,(u!v!)^5}\,n^4 - \frac{1}{3!}\left(\frac{(L_1-1)!}{u!v!}\right)^3 \frac{(2L_1-1)!}{(2u)!(2v)!}\,n^3\right.$$

$$+\left[\frac{1}{2!}\left(\frac{(L_1-1)!}{u!v!}\right)^2 \frac{(3L_1-1)!}{(3u)!(3v)!} + \frac{1}{2!}\left(\frac{(2L_1-1)!}{(2u)!(2v)!}\right)^2 \frac{(L_1-1)!}{u!v!}\right]n^2$$

$$\left.-\left[\frac{(L_1-1)!(4L_1-1)!}{u!v!(4u)!(4v)!} + \frac{(2L_1-1)!(3L_1-1)!}{(2u)!(2v)!(3u)!(3v)!}\right]n + \frac{(L-1)!}{(5u)!(5v)!}\right\},$$

$L = 5L_1$

<u>Type</u> $\left[r^{u}s^{v}\right]^{6}$

$$A = \frac{1}{6!}\left(\frac{(L_1-1)!}{u!v!}\right)^6 n^6 - \frac{1}{4!}\,\frac{(2L_1-1)!}{(2u)!(2v)!}\left(\frac{(L_1-1)!}{u!v!}\right)^4 n^5$$

$$+\left[\frac{1}{3!}\,\frac{(3L_1-1)!}{(3u)!(3v)!}\left(\frac{(L_1-1)!}{u!v!}\right)^3 + \frac{1}{2!2!}\left(\frac{(2L_1-1)!}{(2u)!(2v)!}\right)^2\left(\frac{(L_1-1)!}{u!v!}\right)^2\right]n^4$$

$$-\left[\frac{1}{2!}\,\frac{(4L_1-1)!}{(4u)!(4v)!}\left(\frac{(L-1)!}{u!v!}\right)^2 + \frac{(L_1-1)!(2L_1-1)!(3L_1-1)!}{u!v!(2u)!(2v)!(3u)!(3v)!} + \frac{1}{3!}\left(\frac{(2L_1-1)!}{(2u)!(2v)!}\right)^3\right]n^3$$

$$+\left[\frac{(L_1-1)!(5L_1-1)!}{u!v!(5u)!(5v)!} + \frac{(2L_1-1)!(4L_1-1)!}{(2u)!(2v)!(4u)!(4v)!} + \frac{1}{2!}\left(\frac{(3L_1-1)!}{(3u)!(3v)!}\right)^2\right]n^2$$

$$-\frac{(L-1)!}{(6u)!(6v)!}\,n$$

<u>Type</u> $\left[r^{u}s^{v}\right]^{m}$

$$A = \sum_{j=1}^{m}\sum_{1}\frac{(-1)^{2m-j+1}}{c_1!\cdots c_j!}\left(\frac{(L_1-1)!}{u!v!}\right)^{c_1}\left(\frac{(2L_1-1)!}{(2u)!(2v)!}\right)^{c_2}\cdots\left(\frac{(jL_1-1)!}{(ju)!(jv)!}\right)^{c_j} n^{m-j+1}$$

Where $\sum_1$ is taken over all partitions $\left[1^{c_1}2^{c_2}\ldots j^{c_j}\right]$ of m such that $c_1+c_2+\ldots+c_j = m-j+1$.

Term $a_o^{e_o} a_1^{e_1} a_2^{e_2}$

$$A = \frac{n}{n-e_2}\binom{n-e_2}{e_2}$$

Term $a_o^{e_o}\ a_1^{e_1}\ a_3^{e_3}$

$$A = \frac{n}{e_1 + e_3}\binom{e_1+e_3}{e_1}, \text{ if weight } w = e_1 + 3e_3 = n ,$$

$$A = \frac{n}{n - e_3}\binom{n-e_3}{e_1}\ ; \text{ if } \quad w = 2n$$

Term $a_o^{e_o}\ a_1^{e_1}\ a_4^{e_4}$

$$A = \frac{n}{e_1 + e_4}\binom{e_1+e_4}{e_1}, \quad w = e_1 + 4e_4 = n ,$$

$$A = \frac{n}{n-2e_4}\sum_{x=o}^{x \leq \frac{1}{2}(e_4-1)}\binom{n+x-2e_4-1}{n-2e_4-1}\binom{e_4-2x-1}{n-2e_4-1}, \quad w = 2n \quad ,$$

$$A = \frac{n}{n-e_4}\binom{n-e_4}{e_1}, \quad w = 3n$$

Term $a_o^{e_o}\ a_1^{e_1}\ a_p^{e_p}$

$$A = \frac{n}{n - e_p}\binom{n-e_p}{e_1}, \quad \text{if } w = e_1 + pe_p = (p-1)\, n$$

Term $a_o^{e_o}\ a_1^{e_1}\ a_p$

$$A = n$$

Term $a_o^{e_o}\ a_1^{e_1}\ a_p^{2}$

$$A = \frac{n}{2}\ (e_1 + 1)\ , \quad w = e_1 + 2p = n ,$$

$$A = \frac{n}{2}\ (e_o + 1)\ , \quad w = 2n$$

Term $a_o^{e_o}\, a_1^{e_1}\, a_q^{3}$

$$A = \frac{n}{e_1+3}\binom{e_1+3}{3}, \qquad w = n$$

$$A = \frac{1}{2}(u+1)n^2 + \left[\frac{1}{6}(u+v+2)(u+v+1) + (u+1)(u-2v)\right] n$$

where $w = 2n$, $u = n-q$, $v = n-2q$.

$$A = \frac{1}{e_1!3!}\left\{\binom{e_1}{e}\binom{2e}{e}(e!)^3\ n^3 - 3\binom{e_1}{e}(2e+1)!(e!)n^2 + (e_1+2)!\ n\right\}$$

where $w = 3n$, $e_1 = 3e$

Term $a_o^{e_o}\ a_1^{e_1}\ a_q^{e_q}$ and $w = e_q n$

This term must be of type $\left[1^{n-q}\ q\right]^{eq}$, and formula for $\left[r^u s^v\right]^m$ can be applied.

Term $a_o^{e_o}\ a_p\ a_q^k$

$$A = n\binom{d-1}{m}$$

where $d = (n,q)$, $n = n_1 d$, m = greatest integer in $\frac{k}{n_1}$

The only possible chain-type must be $\left[pq^b\right]\left[q^{n_1}\right]^m$

Term $a_o^{e_o}\ a_p^2\ a_q^k$

There are five possible chain-types:

(a) $\left[p^2\right]\left[q^{n_1}\right]^m$, $d = (n,q)$, $n = n_1 d$.

(b) $\left[p^2q^b\right]\left[q^{n_1}\right]^m$,

(c) $\left[pq^b\right]^2\left[q^{n_1}\right]^m$,

(d) $\left\{\left[p^2q^b\right]\left[q^{n_1}\right]^m, \left[pq^c\right]^2\left[q^{n_1}\right]^r\right\}$

(e) $\left\{\left[p^2\right]\left[q^{n_1}\right]^{m'}, \left[pq^b\right]^2\left[q^{n_1}\right]^r\right\}$

coefficients for these five cases:

(a) $A = \frac{n}{2}\left[\binom{d-2}{m} + (n_1 - 1)\binom{d-2}{m-1}\right]$

(b) $A = \frac{n}{2}\left[(b+1)\binom{d-2}{m} + (n_1 - b - 1)\binom{d-2}{m-1}\right]$

(c) $A = \frac{n}{2}\left[(n_1 - 2b - 1)\binom{d-1}{m} + (n - n_1)\binom{d-2}{m}\right]$

(d) $A = \frac{n}{2}\left[(b+1)\binom{d-1}{m} + (n - n_1)\binom{d-2}{m-1}\right]$

(e) $A = \frac{n}{2}\left[\binom{d-1}{m} + (n - n_1)\binom{d-2}{m-1}\right]$

3. Coefficients of terms involving a_o^{n-r} with $r \leq 6$.

Term	Type	Coefficient
a_o^n		1
$a_o^{n-2} a_p a_q$	$[pq]$	n
$a_o^{n-2} a_p^2$	$[p^2]$	$n/2$
$a_o^{n-3} a_p a_q a_r$	$[pqr]$	$2n$
$a_o^{n-3} a_p^2 a_q$	$[p^2q]$	n
$a_o^{n-3} a_p^3$	$[p^3]$	$n/3$
$a_o^{n-4} a_p a_q a_r a_s$	$[pqrs]$	$6n$
	$[pq]\,[rs]$	$n(n-2)$
$a_o^{n-4} a_p^2 a_q a_r$	$[p^2qr]$	$3n$
	$[p^2]\,[qr]$	$\frac{1}{2}n(n-2)$
$a_o^{n-4} a_p^3 a_q$	$[p^3q]$	n
$a_o^{n-4} a_p^2 a_q^2$	$[p^2q^2]$	$\frac{3}{2}n$
	$[pq]^2$	$\frac{1}{2}n(n-3)$
$a_o^{n-4} a_p^4$	$[p^4]$	$n/4$
	$[p^2]^2$	$\frac{1}{8}n(n-2)$

Type	Term	Coefficient
	$a_o^{n-5}a_pa_qa_ra_sa_t$	
$[pqrst]$		$24n$
$[pq]\ [rst]$		$2n^2$
	$a_o^{n-5}a_p^2a_qa_ra_s$	
$[p^2qrs]$		$12n$
$[ps]\ [pqr]$		$2n(n-4)$
$[p^2s]\ [qr]$		n^2
$[qrs]\ [p^2]$		n^2
	$a_o^{n-5}a_p^3a_qa_r$	
$[p^3qr]$		$4n$
$[p^2r]\ [pq]$		$n(n-4)$
$[p^3]\ [qr]$		$n^2/3$
$[pqr]\ [p^2]$		$n(n-4)$
	$a_o^{n-5}a_pa_q^2a_r^2$	
$[pq^2r^2]$		$6n$
$[qr^2]\ [pq]$		$n(n-4)$
$[pq^2]\ [r^2]$		$\frac{1}{2}n^2$
	$a_o^{n-5}a_p^2a_q^3$	
$[p^3q^2]$		$2n$
$[q^3]\ [p^2]$		$n^2/6$
$[p^2q]\ [q^2]$		$\frac{1}{2}n(n-4)$
	$a_o^{n-5}a_p^4a_q$	
$[p^4q]$		n
$[p^3]\ [pq]$		$\frac{1}{3}n(n-3)$
	$a_o^{n-5}a_p^5$	
$[p^5]$		$n/5$
	$a_o^{n-6}a_{p_1}a_{p_2}a_{p_3}a_{p_4}a_{p_5}a_{p_6}$	
$[p_1p_2p_3p_4p_5p_6] = [123456]$		$120n$
$[p_1p_2][p_3p_4p_5p_6] = [12]\ [3456]$		$6n(n+4)$
$[123][456]$		$4n(n+12)$
$[12][34][56]$		$n(n^2-6n+24)$
$\{[12][3456],\ [134]\ [256]\}$		$n(10n-32)$
$\{[123]\ [456],\ [14]\ [25]\ [36]\}$		$n(n^2-2n-16)$

Term

$a_o^{n-6}\, a_{p_1} a_{p_2} a_{p_3} a_{p_4} a_{p_5}^2$

Type	Coefficient
$[12345^2]$	$60n$
$[12]\,[345^2]$	$3n(n+4)$
$[15]\,[2345]$	$6n(n-4)$
$[1234]\,[5^2]$	$3n(n+4)$
$[125]\,[345]$	$4n(n-1)$
$[15^2]\,[234]$	$2n(n+12)$
$[12]\,[34]\,[5^2]$	$\frac{n}{2}(n^2-6n+24)$
$\{[125]\,[345],\ [13]\,[245^2]\}$	$n(7n-36)$
$\{[15^2]\,[234],\ [12]\,[345^2]\}$	$n(5n-16)$
$\{[15^2]\,[234],\ [25]\,[1345]\}$	$4n(2n-11)$
$\{[125]\,[345],\ [1234]\,[5^2]\}$	$n(7n-36)$
$\{[12]\,[34]\,[5^2],\ [135]\,[245]\}$	$\frac{1}{2}n(n^2+2n-40)$

$a_o^{n-6}\, a_{p_1} a_{p_2} a_{p_3} a_{p_4}^3$

$[1234^3]$	$20n$
$[1234]\,[4^2]$	$n(3n-16)$
$[124^2]\,[34]$	$n(3n-16)$
$[14^3]\,[23]$	$n(n+4)$
$[123]\,[4^3]$	$\frac{2}{3}n(n+12)$
$[14^2]\,[234]$	$2n(n-4)$
$\{[124^2]\,[34],\ [123]\,[4^3]\}$	$\frac{n}{3}(11n-60)$
$\{[14^3]\,[23],\ [24^2]\,[134]\}$	$n(3n-16)$

$a_o^{n-6}\, a_{p_1} a_{p_2} a_{p_3}^4$

$[123^4]$	$5n$
$[12]\,[3^4]$	$\frac{n}{4}(n+4)$
$[13]\,[23^3]$	$n(n-5)$
$[123]\,[3^3]$	$\frac{n}{3}(2n-9)$
$[12]\,[3^2]^2$	$\frac{n}{8}(n-2)(n-4)$

$a_o^{n-6}\, a_{p_1} a_{p_2}^5$

$[12^5]$	n
$[12]\,[2^4]$	$\frac{n}{4}(n-4)$

$a_0^{n-6} a_{p_1}^6$

$[1^6]$	$n/6$
$[1^3]^2$	$\frac{n}{18}(n-3)$
$[1^2]^3$	$\frac{n}{48}(n-2)(n-4)$

$a_0^{n-6} a_{p_1} a_{p_2} a_{p_3}^2 a_{p_4}^2$

$[123^24^2]$	$30n$
$[12][3^24^2]$	$\frac{n}{2}(3n+12)$
$[124^2][3^2]$	$\frac{n}{2}(3n+12)$
$[123][34^2]$	$2n(n-1)$
$[13^2][24^2]$	$n(n+12)$
$[12][34]^2$	$\frac{n}{2}(n-3)(n-4)$
$\{[12][3^24^2], [13^2][24^2]\}$	$\frac{n}{2}(5n-16)$
$\{[13][234^2], [124][3^24]\}$	$n(5n-28)$
$\{[13][234^2], [14^2][23^2]\}$	$2n(2n-11)$
$\{[124^2][3^2], [123][34^2]\}$	$\frac{n}{2}(7n-36)$
$\{[12][34]^2, [13^2][24^2]\}$	$\frac{n^2}{2}(n-5)$

$a_0^{n-6} a_{p_1} a_{p_2}^2 a_{p_3}^3$

$[12^23^3]$	$10n$
$[12][23^3]$	$n(n-4)$
$[13][2^23^2]$	$\frac{n}{2}(3n-16)$
$[2^2][13^3]$	$\frac{n}{2}(n+4)$
$[3^2][12^23]$	$\frac{n}{2}(3n-16)$
$[3^3][12^2]$	$\frac{n}{3}(n+12)$
$[13^2][2^23]$	$n(n-4)$
$\{[12][23^3], [2^23][13^2]\}$	$2n(n-5)$
$\{[13][2^23^2], [12^2][3^3]\}$	$\frac{n}{6}(11n-60)$

$a_0^{n-6} a_{p_1}^2 a_{p_2}^2 a_{p_3}^2$

$[1^22^23^2]$	$15n$
$[123]^2$	$n(2n-11)$
$[12^2][13^2]$	$n(n-1)$
$[1^2][23]^2$	$\frac{n}{4}(n-3)(n-4)$
$\{[1^2][2^23^2], [123]^2\}$	$\frac{n}{4}(11n-60)$
$\{[1^2][23]^2, [12^2][13^2]\}$	$\frac{n}{4}(n-4)(n+1)$

$$\underline{a_o^{n-6} a_{p_1}^2 a_{p_2}^4}$$

$[1^2 2^4]$	$\frac{5}{2} n$
$[12^2]^2$	$\frac{n}{2}(n-5)$
$\{[1^2][2^4], [12^2]^2\}$	$\frac{5}{8} n(n-4)$

$$\underline{a_o^{n-6} a_{p_1}^3 a_{p_2}^3}$$

$[1^3 2^3]$	$\frac{10}{3} n$
$[1^2][12^3]$	$\frac{n}{6}(3n-16)$
$[12]^3$	$\frac{n}{6}(n-4)(n-5)$
$\{[12]^3, [1^3][2^3]\}$	$\frac{n}{18}(n-3)(3n-16)$

4. Some data on I and J functions of general 2-chain terms

(See section 11).

Identities of the J-functions.

(J1) $J_x(i_1, i_2, \dots, i_m; i_1, i_2, \dots, i_m) = 0$

(J2) $J_x(i_1, i_2, \dots, i_k, i_{k+1}, \dots, i_m; j_1, \dots, j_k, j_{k+1}, \dots, j_m)$

$= J_x(i_1, \dots, i_{k+1}, i_k, \dots, i_m; j_1, \dots, j_{k+1}, j_k, \dots, j_m)$

(J3) $J_x(i_1, \dots, i_m; j_1, \dots, j_m) = J_x(i_1', \dots, i_m'; j_1', \dots, j_m')$,

where $(i_1', \dots, i_m')$ is any permutation of $(i_1, \dots, i_m)$, and $(j_1', \dots, j_m')$ is the <u>same</u> permutation of $(j_1, \dots, j_m)$.

(J4) $J_x(i_1, \dots, i_m; j_1, \dots, j_m) = J_x(j_1, \dots, j_m; i_1, \dots, i_m)$

(J5) $J_{x+1}(x, 0, \dots 0, 1, 0, \dots 0; x, 0, \dots, 0, 1, 0, \dots, 0) = 1$

The two 1's must be in different positions.

(J6) $J_{x+1}(x, 0, \dots, 0, 1, 0, \dots, 0; (x+1), 0, \dots, 0) = 1$

(J7) $J_{x+1}(x-1, 0, \dots, 0, 2, 0, \dots, 0; (x+1), 0, \dots, 0) = x$

(J8) $J_x(a, a, \dots, a; j_1, \dots, j_m) = J_x(a, \dots, a; j_1', \dots, j_m')$

Where $(j_1', \dots, j_m')$ is any permutation of $(j_1, \dots, j_m)$.

Some I function identities.

(I1) $I(i_1,\ldots,i_m;r_\alpha,r_\beta) = I(i_1,\ldots,i_m;r_\beta,r_\alpha)$.

(I2) $I(i_1,\ldots,i_p,\ldots,i_q,\ldots,i_m;r_\alpha,r_\beta) = I(i_1,\ldots,i_q,\ldots,i_p,\ldots,i_m;r_\alpha,r_\beta)$,

if $p \neq \alpha,\beta$, and $q \neq \alpha,\beta$.

(I3) $I(i_1,\ldots,i_m;r_\alpha,r_\beta) = I(i'_1,\ldots,i'_m;r_\alpha,r_\beta)$

where $(i'_1,\ldots,i'_m)$ is any permutation of $(i_1,\ldots,i_m)$ that leaves i_α and i_β unchanged.

(I4) $I(i_1,\ldots,i_\alpha,\ldots,i_\alpha,\ldots,i_m;r_\alpha,r_\beta) = I(i_1,\ldots,i_\alpha,\ldots,i_\alpha,\ldots,i_m;r_\alpha,r_\beta)$.

(α may $=\beta$).

(I5) $I(i_1,\ldots,i_m;r_\alpha,r_\beta) = 0$ if either i_α or $i_\beta = 0$

(I6) $\sum\limits_{\alpha,\beta} I(i_1,\ldots,i_m;r_\alpha,r_\beta) = \sum\limits_{\alpha,\beta} I(i'_1,\ldots,i'_m;r_\alpha,r_\beta)$

where $(i'_1,\ldots,i'_m)$ is any permutation of $(i_1,\ldots,i_m)$

(I7) $\sum\limits_{\alpha,\beta} I(\underbrace{a,a,\ldots,a}_{m};r_\alpha,r_\beta) = \binom{m}{2} I(a,\ldots,a;r_1,r_2)$

Because of the symmetry properties listed above it is not necessary to compute all I's. The following list is sufficient for certain of the simpler cases for m = 3 and m = 4.

Values of $I_p\ (i_1, i_2, \ldots, i_m; r_\alpha, r_\beta)$

$m = 3, \quad p = 2$

i_1, i_2, i_3	(1,2)	(1,3)	(2,3)
(2,0,0)	0	0	0
(1,1,0)	1	0	0

$= (r_\alpha, r_\beta)$, e.g., $I_2(1,1,0;1,2) = 1$

$m = 3, \quad p = 3$

	(1,2)	(1,3)	(2,3)
(3,0,0)	0	0	0
(2,1,0)	1	0	0
(1,1,1)	3	3	3

$m = 3, \quad p = 4$

	(1,2)	(1,3)	(2,3)
(4,0,0)	0	0	0
(3,1,0)	1	0	0
(3,0,1)	0	1	0
(1,3,0)	1	0	0
(1,0,3)	0	1	0
(0,1,3)	0	0	1
(0,3,1)	0	0	1
(2,1,1)	10	10	2
(1,2,1)	10	2	10
(1,1,2)	2	2	2
(2,2,0)	3	0	0
(2,0,2)	0	3	0
(0,2,2)	0	0	3

$m = 3, \quad p = 5$

	(1,2)	(1,3)	(2,3)
(5,0,0)	0	0	0
(4,1,0)	1	0	0
(4,0,1)	0	1	0
(1,0,4)	0	1	0
(1,4,0)	1	0	0
(0,1,4)	0	0	1
(0,4,1)	0	0	1
(0,2,3)	0	0	6
(0,3,2)	0	0	6
(2,0,3)	0	6	0
(2,3,0)	6	0	0
(3,0,2)	0	6	0
(3,2,0)	6	0	0
(2,2,1)	65	32	32
(2,1,2)	32	65	32
(1,2,2)	32	32	65

$m = 3 \quad p = 6$

	(1,2)	(1,3)	(2,3)
(6,0,0)	0	0	0
(5,1,0)	1	0	0
(4,2,0)	10	0	0
(3,3,0)	34	0	0
(4,1,1)	27	27	1
(3,2,1)	225	117	72
(2,2,2)	383	383	383

m = 4

	(1,2)	(1,3)	(1,4)	(2,3)	(2,4)	(3,4)
(1,1,0,0)	1	0	0	0	0	0
(1,1,1,0)	3	3	0	3	0	0
(1,1,1,1)	26	26	26	26	26	26
(2,1,0,0)	1	0	0	0	0	0
(2,1,1,0)	10	10	0	5	0	0
(2,1,1,1)	178	178	178	89	89	89